# Pricing Default Events: Surprise, Exogeneity and Contagion

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- When investors are averse to a given risk, a security whose payoffs are exposed to this risk are less valuable than those whose payoffs are not.
- A defaultable bond exposes its holder to two risks:
  - **(a)** the risk that future probabilities of default change,
  - **(b)** the risk that the bond issuer effectively defaults.
- In order to derive closed form expressions of the prices of credit derivatives, most reduced-form models of credit risk ”price” risk (a) but not the default events themselves (risk (b)).
- That is, they implicitly consider that investors are not averse to the default-event surprise (or that these surprises can be diversified away).

- A few papers mention this approximation and try to take into account the surprise, e.g.:
  - Jarrow, Yu (2001, JoF) ["Counterparty Risk and the Pricing of Defaultable Securities"]. For 2 debtors only.
  - A series of paper by Bai, Collin-Dufresne, Goldstein, Helwege (2013), with a very specific modeling of default dependence.
- In general:
  - Default dependence is difficult to specify.
  - Derivative prices have no closed-form expressions.

- This paper solves this problem for credit derivatives (CDS and CDO) written on a pool of credits, which can be partitioned into $J$ ”large” homogenous segments.
- The model accommodates different forms of contagion:
  - exposure to common factors (frailty);
  - self-exciting defaults;
  - contagion across sectors.
- Based on U.S. bond data, an application illustrates that this feature provides an explanation for the so-called *credit-spread puzzle*.

## Outline of the presentation

1. Introduction.
2. The standard reduced-form approach and its limitations.
3. Modeling Framework and Derivative Pricing.
4. Applications.

# 2. The Standard Approach and its Limitations

## Notations

- A pool of $I$ entitites $i = 1, \ldots, I$.
- Default indicators $d_{i,t}$:

$$d_{i,t} = \begin{cases} 1 & \text{if entity } i \text{ is in default at date } t, \\ 0 & \text{otherwise.} \end{cases}$$

- $n_t$ the number of defaults occurring at date $t$.
- $N_t = \sum_{\tau=1}^{t} n_\tau$ the cumulated number of defaults.

## Assumptions on the historical distribution

i) **Homogenous portfolio**
The default indicators $d_{i,t+1}$ are independent, identically distributed given $\underline{F_{t+1}}, \underline{d_t}$.

ii) **Default dependence driven by** $F$
This conditional distribution depends on factor $F_{t+1}$ only.

iii) $F$ **is exogenous**
The conditional distribution of $F_{t+1}$ given $(\underline{F_t}, \underline{d_t})$ is equal to the conditional distribution of $F_{t+1}$ given $F_t$.

Remark: (i) and (ii) will be relaxed in our general model.

## The standard pricing approach

- Assumption on the stochastic discount factor:

$$\tilde{m}_{t,t+1} = \tilde{m}(F_{t+1}).$$

- Then the price of a payoff $g(N_{t+h})$ at date $t$ is:

$$\begin{aligned} \tilde{\Pi}(g,h) &= E_t[\tilde{m}_{t,t+1} \dots \tilde{m}_{t+h-1,t+h} g(N_{t+h})] \\ &= E_t[\tilde{m}_{t,t+1} \dots \tilde{m}_{t+h-1,t+h} \tilde{g}(\underline{F_{t+h}})], \end{aligned}$$

where : $\tilde{g}(\underline{F_{t+h}}) = E[g(N_{t+h})|\underline{F_{t+h}}]$.

- Therefore : $\tilde{\Pi}(g,h) = \tilde{\Pi}(\tilde{g},h)$.

$\Rightarrow$ **It is equivalent to price $g(N_{t+h})$ or to price its prediction $\tilde{g}(\underline{F_{t+h}})$.**

## Risk premia associated with default events

- What is the change in pricing formula, when $m_{t,t+1} = m(F_{t+1}, n_{t+1})$?
- Let us consider the projected sdf:

$$\tilde{m}_{t,t+1} = E[m(F_{t+1}, n_{t+1})|F_{t+1}].$$

- Then:

$$\Pi(g,h) = \underbrace{\tilde{\Pi}(g,h)}_{\text{standard formula}} + \underbrace{\Pi(g-\tilde{g},h)}_{\text{the price of surprise}}.$$

**standard formula** **the price of surprise**

## Relaxing the exogeneity assumption

- New assumption: The conditional historical distribution of $F_{t+1}$ given $\underline{F}_t, \underline{d}_t$ is equal to the conditional distribution of $F_{t+1}$ given $\underline{F}_t, n_t$.
- A more complete decomposition of the derivative price :

$$\Pi(g,h) = \tilde{\Pi}(g,h) + [\Pi(\tilde{g},h) - \tilde{\Pi}(\tilde{g},h)] + \Pi(g-\tilde{g},h)$$

| price | = | standard price | + | causality adjustment | + | surprise adjustment |
|---|---|---|---|---|---|---|

- Moreover, we show that the standard formula for pricing a corporate zero-coupon bond:

$$\begin{aligned} B(t,h) &= \left(E_t^Q[\exp(-r_t \ldots - r_{t+h-1})\mathbf{1}_{d_{i,t+h=0}}]\right) \\ &= E_t^Q[\exp(-r_t \ldots - r_{t+h-1} - \lambda_{t+1}^Q \ldots - \lambda_{t+h}^Q)], \end{aligned}$$

cannot be used in general.

## Default intensities

- If $\Omega_t^* = (\underline{F_{t+1}}, \underline{d_t})$, the historical intensity $\lambda_{t+1}$ is defined by:

$$P(d_{t+1} = 0 | d_t = 0, \Omega_t^*) = \exp(-\lambda_{t+1}).$$

- The risk-neutral intensity $\lambda_{t+1}$ is defined by:

$$Q(d_{t+1} = 0 | d_t = 0, \Omega_t^*) = \exp(-\lambda_{t+1}^Q),$$

- If $m_{t,t+1} = \exp(\delta_0 + \delta_F' F_{t+1} + \delta_S n_{t+1})$, the risk-neutral intensity is:

$$\lambda_{t+1}^Q = \lambda_{t+1} + \log\{\exp(-\lambda_{t+1}) + [1 - \exp(-\lambda_{t+1})]\exp(\delta_S).\}$$

# 3. Modeling Framework and Derivative Pricing

- To get (quasi) closed form expressions for derivative prices, we need affine processes.
- The joint process $(d_{1t}, \ldots, d_{It}, F_t)$ cannot be affine, but the aggregate process $(n_t, F_t)$ can be if the size of the homogenous pool is large.

## Assumptions

**(a)** A Poisson regression model for the default count:

$$n_{t+1} | \underline{F_{t+1}}, \underline{n_t} \sim \mathcal{P}(\beta_F' F_{t+1} + \beta_n n_t + \gamma);$$

**(b)** The conditional Laplace transform of $F_{t+1}$ given $\underline{F_t}$ is exponential affine in $(F_t, n_t)$:

$$E_t[\exp(v' F_{t+1})] = \exp[a_F(v)' F_t + a_n(v)' n_t + b(v)];$$

**(c)** The s.d.f. is exponential affine in both $F_{t+1}$ and $n_{t+1}$:

$$m_{t,t+1} = \exp(\delta_0 + \delta_F' F_{t+1} + \delta_S n_{t+1}).$$

- In that setup, $(F_t, n_t)$ is jointly affine.
- Then the price at date $t$ of any exponential payoff $\exp(uN_{t+h})$ can be derived by recursion.
- Since the pool is homogenous, we know also how to price:
  - individual default $d_1$ (single name CDS),
  - joint defaults $d_1 d_2 \dots$.
- Indeed:

$$\Pi(d_1 \dots d_K, h) = \frac{1}{I(I-1)\dots(I-K+1)} \left[ \frac{d^K}{dv^K} \Pi(\exp(N \log v), h) \right]_{v=1}.$$

- The price of a non-exponential payoff deduced by Fourier transform [Duffie, Pan, Singleton (2000)]: CDO pricing, tranches.

## Extension: Heterogeneous pools

- The pool can be partitioned into $J$ homogenous pools, with different risk characteristics.
- For corporations, the segment can be defined by the industrial sector, by the size, by the domestic country, but the rating cannot be used since it is time-varying.
- $n_{j,t}, j = 1, \ldots, J$ denote the numbers of defaults in each segment, conditionally independent :

$$n_{j,t+1} \sim \mathcal{P}[\beta_j' F_{t+1} + C_j' n_t + \gamma_j], j = 1, \ldots, J.$$

$\Rightarrow$ Additional contagion channel: across sectors.

# 4. Illustrations

## App.1: Contagion and network

An illustration with six homogenous segments of size 100.

- Two types of factors:

  $F_{B,t}$ a sequence of i.i.d. Bernoulli variables

  $F_{N,t}$ processes keeping memory of past default counts in each segment

  $$F_{N,j,t} = \rho F_{N,j,t-1} + n_{j,t-1}, j = 1, \ldots, 6.$$

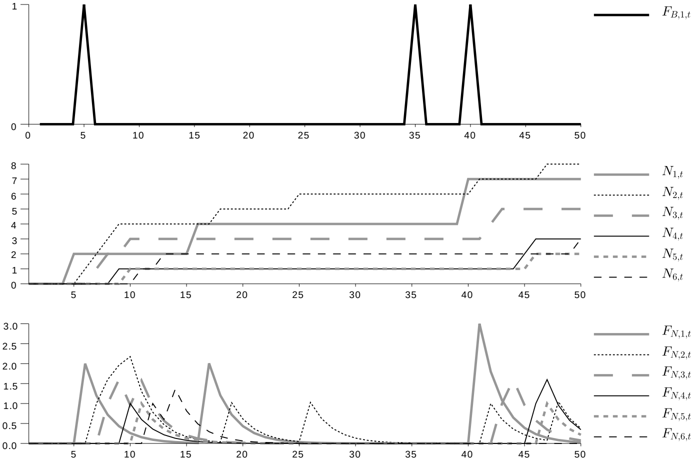
- The distribution of the count variables with a circular structure of the network:

$$n_{1,t+1} \sim \mathcal{P}(0.4 F_{N,6,t} + F_{B,t}), n_{j,t+1} \sim \mathcal{P}(0.4 F_{N,j-1,t}), j = 2, \ldots, 6.$$

## App.1: Contagion and network

- The next figure gives the evolutions of factors and default counts.
- A high value of factor $F_B$ may immediately generate defaults in segment 1.
- These defaults propagate to the other segments by contagion.

# App.1: Contagion and network



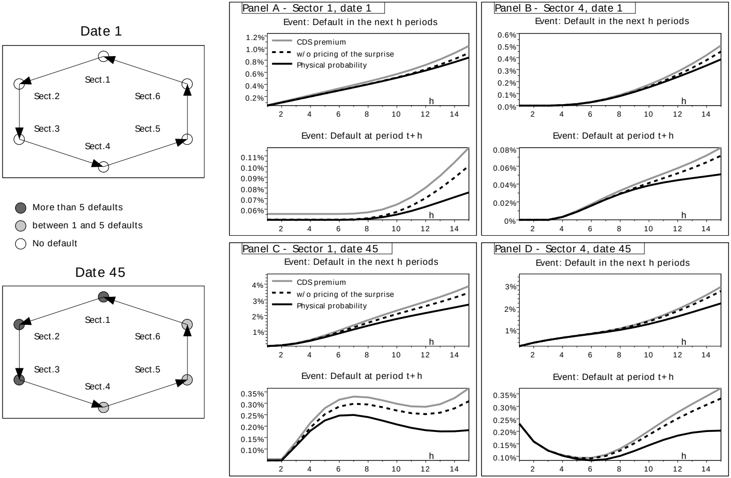
## App.1: Contagion and network

The next figure displays the term structures of:

- the CDS premium,
- the CDS without pricing the surprise,
- the actuarial value (physical probability)

for two dates and segments.

# App.1: Contagion and network



## App.2: Credit Spread Puzzle

- Credit-spread puzzle: observation of a wide gap between
  - **(a)** Credit Default Swap (CDS) spreads, that can be seen as default-loss expectations under the risk-neutral measure, and
  - **(b)** expected default losses (under $P$).
  - $\Rightarrow$ See e.g. D'Amato, Remolona (2003), Hull, Predescu, White (2005).
- Standard credit-risk models, that do not price default-event surprises, deal with the credit-risk puzzle by incorporating credit-risk premia. But these premia are too small for short maturities.
- We show that pricing default-event surprises may solve the credit-puzzle for all maturities, including the shortest ones.

## App.2: Credit Spread Puzzle

- We calibrate our model on U.S. banking-sector bond data covering the last two decades.
- Specifically, we consider riskfree (Treasury) bonds and bonds issued by U.S. banks (1995-2013), rated BBB.
- Our results suggest that neglecting the pricing of default events is likely to result in an overestimation of model-implied physical probabilities of defaults for short-term horizons.

## App.2: Credit Spread Puzzle

| | $\delta_{F,1}$ | $\delta_{F,2}$ | $\delta_{F,3}$ | $\delta_{F,4}$ | $\delta_S$ | $\delta_0$ |
|---|---|---|---|---|---|---|
| M1 | 1 | -0.974 | 3.045 | -5.063 | 1.163 | -0.044 |
| M2 | 1 | -0.972 | 5.681 | -5.589 | - | -0.081 |
| | $\mu_1$ | $\nu_1$ | $\rho_1$ | $\mu_2$ | $\nu_2$ | $\rho_2$ |
| M1 | 1.55 | 0.022 | 0.95 | 0.428 | 0.004 | 0.95 |
| M2 | 3.26 | 0.021 | 0.95 | 0.267 | 0.006 | 0.95 |

- M1 (resp. M2) is the model pricing the default-event surprise, i.e. with $\delta_S \neq 0$ (resp. $\delta_S = 0$).
- $F_{1,t}$ and $F_{2,t}$ follow independent ARG processes [$(\mu_1, \rho_1, \nu_1)$ and $(\mu_2, \rho_2, \nu_2)$, respectively].
- The sdf is given by $m_{t,t+1} = \exp(\delta_0 + \delta_F' F_{t+1} + \delta_S n_{t+1})$ where $F_t = [F_{1,t}, F_{1,t-1}, F_{2,t}, F_{2,t-1}]'$.
- The conditional distribution of $n_t$ given $\underline{F_t}, \underline{n_{t-1}}$ is Poisson $\mathcal{P}(F_{2,t})$.
- Calibration is carried out to reproduce a set of unconditional moments derived from observed data (fitted moments on next slide).

# App.2: Credit Spread Puzzle

**Panel A - Unconditional moments (means / standard deviations)**
S: Sample, M1: model pricing the surprise, M2: model not pricing the surprise.

| | Treasuries (riskfree) yields | | | | Spreads (Banks vs. Treas.) | | | Correlations | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 mth | 1y | 3y | 5y | 1y | 3y | 5y | 1y | 3y | 5y |
| $\omega$ | 50 | 50 | 50 | 50 | 100 | 100 | 100 | 0.05 | 0.05 | 0.05 |
| S | 2.7/2.1 | 3.1/2.2 | 3.5/2.0 | 3.9/1.8 | 2.0/1.6 | 2.5/1.8 | 2.8/2.0 | -60 | -70 | -65 |
| M1 | 2.7/2.2 | 3.1/2.1 | 3.6/1.9 | 3.9/1.8 | 1.7/1.9 | 2.3/1.8 | 3.1/1.8 | -65 | -65 | -65 |
| M2 | 2.6/1.7 | 3.1/1.7 | 3.8/1.8 | 3.8/2.3 | 0.6/0.9 | 1.3/1.3 | 2.6/2.4 | -47 | -54 | -70 |

**Panel B - Time-series fit (MSE divided by series variances, in %)**

| | Treasuries (riskfree) yields | | | | Spreads (Banks vs. Treas.) | | |
|---|---|---|---|---|---|---|---|
| | 1 mth | 1y | 3y | 5y | 1y | 3y | 5y |
| M1 | 8.6 | 2.8 | 0.2 | 3.1 | 11.1 | 1.2 | 7.2 |
| M2 | 16.3 | 9.2 | 1.0 | 36.3 | 57.8 | 19.5 | 24.2 |

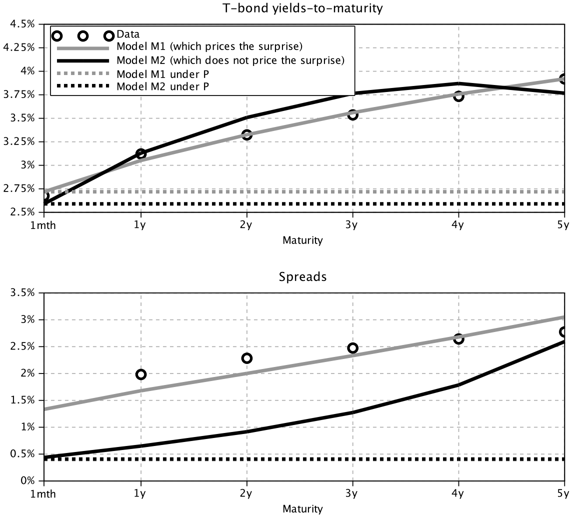
- M1 and M2 are estimated by weighted-moment methods (weights provided in row $\omega$).

$\Rightarrow$ Model M1 is better than M2 at reproducing sample moments, especially at the short-end of the term structure of spreads.

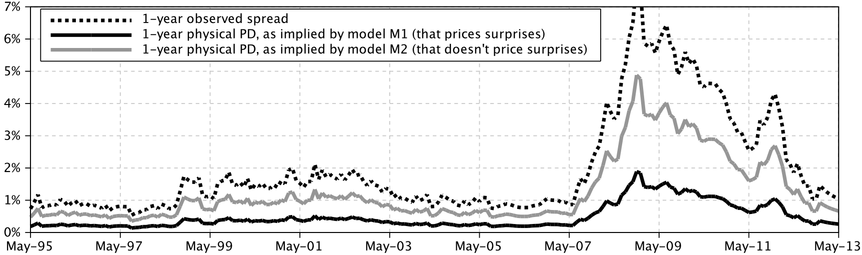
- Panel B reports the ratios of mean squared pricing errors (MSE) to the sample variances of corresponding yields/spreads.

$\Rightarrow$ Pricing errors obtained with M1 are far lower than those associated with M2.

# App.2: Credit Spread Puzzle



# App.2: Credit Spread Puzzle



# 6. Conclusion

- Standard approaches of credit-risk pricing neglect default-event surprises.
- This paper proposes a tractable way to price these surprises.
- In our framework, quasi-closed-form expressions for derivative prices still exist if the sizes of the homogenous segments are sufficiently large.
- The specification accommodates different forms of contagion.
- An empirical analysis suggests that models pricing default-event surprises can generate sizable credit-risk premia at the short end of the yield curve and, hence, can solve the credit-risk puzzle.

Appendix

- If $F_t$ is exogenous under $P$ and $\delta_S \neq 0$, $F_t$ is no longer exogenous under $Q$.
- The intensity $\lambda_{i,t}$ is a pre-intensity if:

$$P(\tau_i > t + h | \tau_i > t, \Omega_t^*) = E\left(\prod_{k=1}^{h} \exp(-\lambda_{i,t+k}) | d_{i,t} = 0, \Omega_t^*\right)$$

  with $\tau_i = \inf\{t : d_{i,t} = 1\}$.
- If $F_t$ is exogenous (under $P$), then $\lambda_{i,t+1}$ is a pre-intensity.
- If $\delta_S \neq 0$, $F_t$ is not exogenous under $Q$ (even if it is exogenous under $P$) then $\lambda_{i,t+1}^{Q}$ is not a pre-intensity, and the standard formula for $B(t,h)$ is not valid.

In fact, the pricing formula becomes:

$$B(t,h) = \overset{Q}{E_t}\left[\exp(-r_t \ldots - r_{t+h-1} - \tilde{\lambda}^{Q}_{t+1,t+h} \ldots - \tilde{\lambda}^{Q}_{t+h,t+h})\right],$$

where

$$\tilde{\lambda}^{Q}_{t+1,t+h} = -\log Q(d_{t+1} = 0 | d_t = 0, \underline{F_{t+h}})$$

is doubly indexed, with the interpretation of a "forward" intensity.

**Homogenous model**

• Factor: $F_t = (F_{1,t}, F_{1,t-1}, F_{2,t})$,
where $(F_{1,t})$ and $(F_{2,t})$ are independent Autoregressive Gamma (ARG) processes.

A lagged value of $F_1$ is introduced to get more flexible specifications of the s.d.f. and of the term structure of the yields.

• Parameter $\beta$ is set in order to get : $n_t | F_t \sim \mathcal{P}(F_{2,t})$

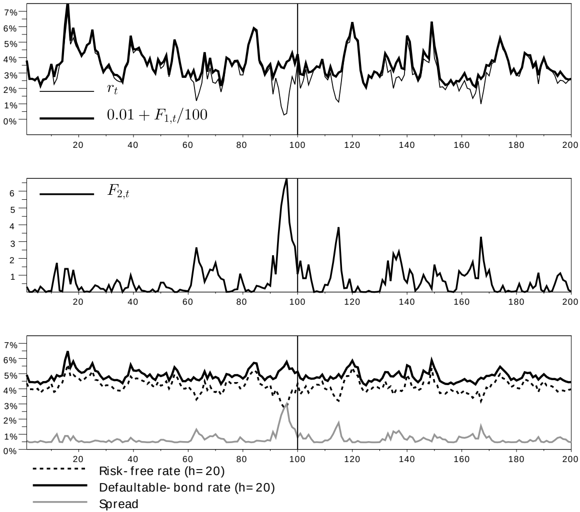
The short term rate is:

$$r_t = K_0 + K_F' F_t,$$

where the coefficients $K_0, K_F$ depend on the parameters characterizing the ARG dynamics, on the $\beta$, and on the parameter in the s.d.f. to ensure the AAO.

The next figure provides the evolutions of:

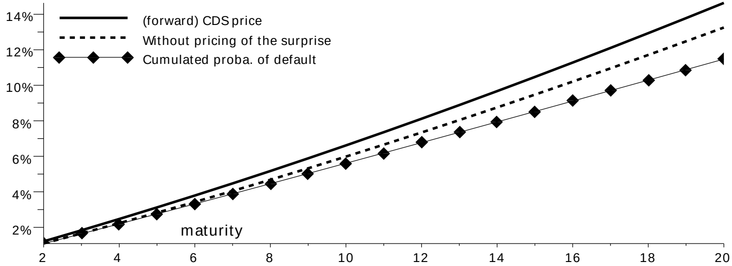
- the factors $F_1, F_2$,
- the short-term rate,
- the defaultable bond rate for the maturity $h = 20$,
- the spread between the latter and its ”riskfree” counterpart (same maturity).


7%
6%
5%
4%
3%
2%
1%
0%
$r_t$
$0.01 + F_{1,t}/100$
20
40
60
80
100
120
140
160
180
200
$F_{2,t}$
6
5
4
3
2
1
Risk-free rate (h=20)
Defaultable-bond rate (h=20)
Spread

- The next figure compares:
  - the (forward) CDS price,
  - this price without pricing the surprise,
  - the cumulated probability of default.
- (Forward) CDS prices to avoid the discounting effects that are implicit in the standard CDS pricing formula.
- Note that the forward CDS prices are not exactly equal to the risk-neutral probability of default.

$\Rightarrow$ **About half of the total credit-risk premia are accounted for by the credit-event risk premia.**

$\Rightarrow$ **This proportion weakly depends on the time-to-maturity.**


14%
12%
10%
8%
6%
4%
2%
(forward) CDS price
Without pricing of the surprise
Cumulated proba. of default
maturity
2
4
6
8
10
12
14
16
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20