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Staying at zero with affine processes: an application to term structure modelling

Alain Monfort
Banque de France and CREST

Fulvio Pegoraro
Banque de France, ECB and CREST

Jean-Paul Renne
HEC Lausanne

Guillaume Roussellet
McGill University

The recent financial crises observed in the United States, the United Kingdom and the euro area have led their respective central banks to bring policy rates down to unprecedented low levels, with an associated dramatic drop of their yield curves. Short-term rates have remained at their lower bound for extended periods of time while longer-term rates have fluctuated with relatively high volatilities. This paper describes a new class of non-negative affine term structure models introduced by Monfort et al. (2017) and used to replicate these features of the yield curve. The proposed empirical analysis also suggests that ignoring interest rate risk premia implies a substantial underestimation of the length of the zero lower bound regime.

Modelling yield curves at the lower bound

Before the burst of the 2008 financial crisis, the Bank of Japan was the only large central bank that had brought its policy rate down close to zero. Since 2010 however, keeping policy rates close to the zero lower bound (ZLB) has become a common practice for the American Federal Reserve System (Fed), the European Central Bank (ECB) and the Bank of England (BoE). In June 2014, the ECB became the first major central bank to lower one of its key policy rates (the deposit facility rate) into negative territory.[1] In all of these currency areas, sharp decreases in short-term rates have pushed the yield curves down to unprecedented low levels.

For instance, between January 1995 and December 2007, the average level of the German sovereign yield curve[2] was between 3% and 5% for 80% of the time (see Chart 1); over the same period, the 3-month rate of the Bund was between 2% and 4% for 80% of the time. Between January 2008 and February 2017, the average German yield curve level was below 1% for 60% of the time, and below 2.5% (see Chart 1) for 90% of the time. The 3-month Bund rate stayed around zero (between –50bps and +50bps) for 75% of the time.

More precisely, we have observed short rates lingering at the lower bound for extended periods of time, while the volatility of longer-term interest rates has not declined and is maturity-dependent. This point is illustrated in the case of Japan in Charts 2 and 3.

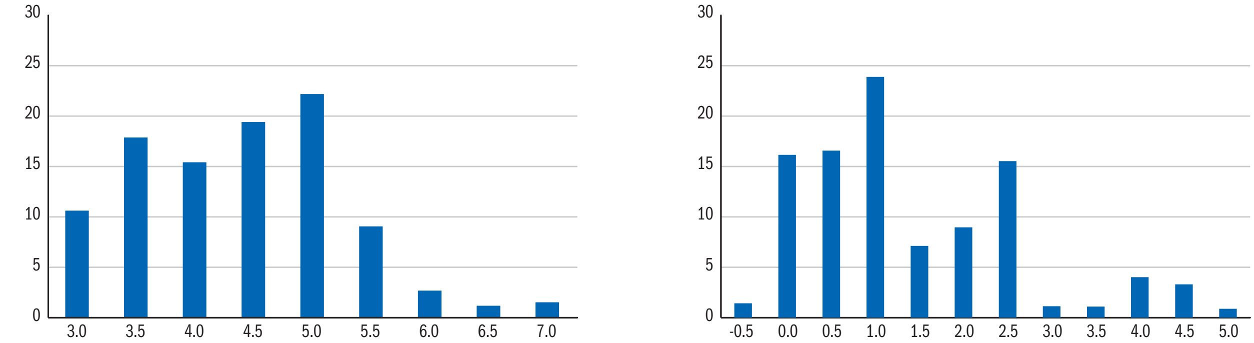
1 *The Sveriges Riksbank adopted negative interest rates in 2009 and 2010, and the Danmarks Nationalbank in 2012. More recently, the Swiss National Bank decided to introduce negative interest rate at the beginning of 2015, followed by the Bank of Japan at the beginning of 2016.*

2 *The average level of the yield curve at a given date is given by the empirical average of the observed yields across the maturity spectrum.*

C1 Average level of the Bund yield curve

(x-axis: percentage rate in annualized basis - 50 basis points intervals; y-axis: frequency of observations)

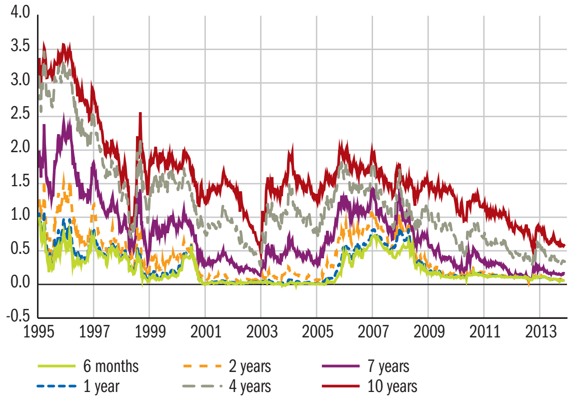
a) between January 1995 and December 2007 — b) between January 2008 and February 2017



Source: Bloomberg.

C2 Japanese government bond (JGB) yield data

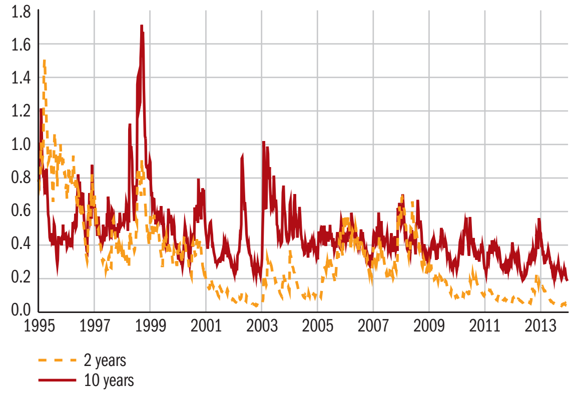
(yields in %, annualized basis)



Source: Bloomberg

C3 Conditional volatility proxies of JGB yields

(yields in %, annualized basis)



Sources: Bloomberg, authors' calculations.

In this unconventional monetary policy context, it is useful to provide a term structure model able to replicate and explain yield curve dynamics characterised by these new empirical features which previous models were not able to match (see Box).

A new affine term structure model

Monfort et al. (2017) introduce a new dynamic term structure model able to simultaneously capture the following characteristics:

(i) yields at all maturities evolving above a lower bound and featuring time-varying (stochastic) conditional variances;

(ii) affine yield-to-maturity formulas;

(iii) extended periods of short-term rates stuck at their lower bound;

(iv) closed-form formulas for lift-off dates.

Such a large degree of tractability and flexibility has been reached thanks to the introduction of a new non-negative affine process (featuring the new and so-called Gamma-zero distribution) entailing a short-rate conditional distribution with a point mass at the relevant (selected) lower bound. The short rate is thus an affine process that can take the lower bound value with a strictly positive probability and yields at longer maturities will also be affine and evolve above this lower bound (see Monfort et al., 2017, for further details). Observe that this term structure model determines the yield-to-maturity formula (model-implied interest rates at any maturity) following the absence of arbitrage opportunity principle.

In addition, our pricing methodology takes into account the risk aversion of investors without affecting the tractability of our pricing formulas: any selected state variable (i.e., the information used by investors to price assets) is seen as a possible source of risk that has to be properly assessed in order to determine interest rates over time. In other words, for any of these factors we detect an associated risk premium that formalises the compensation that a risk averse agent asks for to buy a long-term bond instead of rolling over a short-term (risk-free) one.

The affine nature of our model paves the way for a large degree of flexibility at the estimation stage. First, assuming that factors are unobservable, the model can be written in a State-Space form and the estimation procedure turns out to be computationally simple (based on Kalman filtering techniques). Second, it implies that yield forecasts and conditional variances are known affine functions of the factors. This feature enables us to add (at the estimation stage) these two sets of relationships as extra measurement equations in the State-Space form of the model. Surveys of professional forecasters are used as proxies for the (properly selected) model forecasts, and we approximate the conditional variance of the yields of interest with a GARCH model. Including these equations helps introduce interest rate persistence in the historical factor dynamics and enables the model to replicate the observed interest rate volatilities across the maturity spectrum (see following section).

Arguments in favour of modelling the yield curve

Four main reasons justify the economic and financial relevance of such modelling.

*First, any long-term interest rate can be decomposed into the expected (long-term) path of the short-term rate (also referred to as the expectation component) and a term premium component. The first term tells us what market expectations are about future short rates, while the second one measures the compensation required by investors for bearing interest rate risk. Given that both components are not observable, a model has to be used to implement such a breakdown (**forecast reason**). Second, a model of the yield curve helps study how movements at the short end induced by a monetary policy shock result in longer-term (aggregate-demand-relevant) yields (**monetary policy reason**). Third, when issuing new debt, governments have to decide on the maturity of the new bonds and need to understand the associated impact on the yield curve and public debt (**debt policy reason**). Fourth, investors need yield curve models to properly price and hedge fixed income derivatives like swaps, caps and floors, futures, and options on interest rates (**pricing-hedging reason**).*

The most popular Gaussian affine models do not provide a lower bound (LB) to the risk-free short rate and model-implied yields have constant conditional variances (see Adrian et al., 2013, and Joslin et al., 2011). The class of positive affine models is able to guarantee strictly positive yields featuring time-varying (stochastic) variances but the short rate cannot stay at the ZLB (or any other LB; see Cox et al., 1985, Dai and Singleton, 2003, and references therein). The recently proposed class of shadow rate models (see Bauer and Rudebusch, 2015, Christensen and Rudebusch, 2016, Carriero et al., 2016) introduces a lower bound in an otherwise Gaussian short rate dynamics. Because of this truncation, the yield-to-maturity formula is not explicit anymore. In addition, when the economy is away from the LB (i.e. when the short rate is higher than the truncation level), the state variables feature Gaussian dynamics, thus implying in this case a constant interest rate conditional variance.

Empirical assessment: the case of the Japanese yield curve

Since the Japanese economy has experienced extremely low interest rates since the mid-1990s, the sovereign term structure of Japanese interest rates is a relevant data set for assessing the ability of this new yield curve model to handle the empirical features mentioned above (see Kim and Singleton, 2012).

Monfort et al. (2017) consider end-of-week observations on zero-coupon Japanese government bond (JGB) yields from 16 June 1995 to 30 May 2014, with maturities from 6 months to 10 years. Chart 2 clearly shows that, from 2001 to 2006, the Japanese yield curve entered a ZLB regime with the 6-month interest rate stable at a level virtually equal to zero. We also observe that, during this same period, longer-term rates showed a high degree of maturity-dependent variability, as confirmed by Chart 3.

Monfort et al. (2017) select a model with four unobservable factors: the proposed empirical analysis shows that this new term structure model is able to simultaneously match yield levels and conditional volatilities across the maturity spectrum, and to correctly replicate interest rate surveys. The comparison with alternative specifications highlights the importance of adding extra measurement equations to help the model match the statistical properties of key interest rates.

A very useful outcome of the model is that it provides two estimates of the probabilities of staying in a ZLB regime (or of leaving it, referred to as lift-off probabilities). First, an estimate that takes the point of view of a risk averse investor (typically referred to as historical probabilities); and second, an estimate of risk neutral probabilities which would be obtained if investors were risk neutral, i.e. they did not ask for a compensation for interest rate risk.

Interestingly, we find that ignoring the existence of interest rate risk premia implies a substantial underestimation of the persistence (the length) of the ZLB regime. In other words, the short-term interest rate is expected to remain in a ZLB regime for a longer period of time if we properly take the point of view of a risk averse investor. In particular, we find that the historical probabilities of being in a low-rate environment are in many cases two to three times larger than the risk neutral ones. This result highlights the importance of having a term structure model with affine and well-specified historical and risk neutral dynamics which enables us to easily calculate and compare historical and risk neutral probabilities of staying (leaving) the ZLB.

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