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AN ECONOMETRIC FRAMEWORK**

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# Default, liquidity and crises: an econometric framework\*

Alain Monfort<sup>†</sup>     Jean-Paul Renne<sup>‡</sup>

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**Résumé:** Cet article présente un cadre économétrique général visant à modéliser conjointement les fluctuations de courbes de taux associées à différents émetteurs obligataires. Les titres sous-jacents à ces courbes peuvent différer en termes de qualité de crédit de l'émetteur et/ou en termes de liquidité. Les facteurs de risque sont affectés par des chocs gaussiens dont les covariances dépendent du régime qui prévaut au moment du choc. Les tendances des facteurs de risque dépendent également des régimes. Le processus suivi par les régimes est une chaîne de Markov dont les probabilités de transition sont non-homogènes sous la mesure historique. Bien que riche, le modèle permet de valoriser les obligations à partir de simples formules récursives. Plusieurs exemples numériques sont présentés. En particulier, nous montrons comment les changements de régime peuvent être mis à profit pour modéliser des phénomènes de contagion sectorielle. Une extension visant à intégrer les notes attribuées par les agences de notation est également proposée.

**Classifications JEL:** E43, E44, E47, G12, G24.

**Mots clés:** risque de crédit, risque de liquidité, structure par terme des taux d'intérêt, modèle affine, changement de régime, processus Car.

**Abstract:** In this paper, we present a general discrete-time affine framework aimed at jointly modeling yield curves associated with different debtors. The underlying fixed-income securities may differ in terms of credit quality and/or in terms of liquidity. The risk factors follow conditionally Gaussian processes, with drifts and variance-covariance matrices that are subject to regime shifts described by a Markov chain with (historical) non-homogenous transition probabilities. While flexible, the model remains tractable. In particular, bond prices are given by quasi-explicit formulas. Various numerical examples are proposed, including a sector-contagion model and credit-rating modeling.

**JEL codes:** E43, E44, E47, G12, G24.

**Keywords:** credit risk, liquidity risk, term structure, affine model, regime switching, Car process.

# 1. Introduction

Though already strong before the recent financial crisis, the case for including regime shifts within term-structure models for defaultable bonds is obviously stronger now (see, amongst many others, Christensen, Lopez and Rudebusch, 2009 [16]). This paper proposes a general affine term-structure framework aimed at jointly modeling several yield curves associated with different obligors, in the presence of regime switching.

Motivated by derivative-pricing or credit-risk-management objectives, a large strand of the literature related to fixed-income securities has focused on the joint modeling of several yield curves. In this context, Jarrow, Lando, Turnbull (1997) [45], Lando (1998) [50] or Duffie and Singleton (1999) [28] have highlighted the potential of affine term-structure frameworks to jointly model yield curves associated with various obligors subject to default risk. Their intensity-based –or reduced-form– approaches used to model defaults differ from the more structural approaches originating in Black and Scholes (1973) [10] and Merton (1974) [55].<sup>1</sup> Whereas the intensity-based framework was originally designed to account for default risk, it is also appropriate to model liquidity-pricing effects and it can accommodate time-varying recovery rates as well (see Duffie and Singleton (1999) [28]). Numerous further developments have illustrated the flexibility and tractability of affine-term structure models to capture the comovements of different yield curves (see e.g. Duffee, 1999 [25] or Gourieroux, Monfort and Polimenis, 2006 [38]).

There is strong evidence of regime switching in the dynamics of interest rates (see, e.g., Hamilton, 1988 [41] or Cai, 1994 [11]). Regime shifts have been successfully introduced in term-structure models of risk-free interest rates by, amongst others, Bansal and Zhou (2002) [5], Monfort and Pegoraro (2007) [58], Dai, Singleton and Yang (2007) [19] or Ang Bekaert and Wei (2008) [3]. Whereas these contributions put forward the importance of modeling regime switching in yield-curve models, a few has been done to integrate such a feature in term-structure models of defaultable bonds. However, empirical studies point to the existence of different regimes in the default risk valuation (see, e.g., Davies, 2004 [22] and 2008 [23] or Alexander and Kaeck, 2008 [1]). From a theoretical point of view, Hackbarth, Miao and Morellec (2006) [40] provide a theoretical model to explain the dependence of credit spread on business-cycle regimes. In the same vein, Bhamra, Kuehn and Strebulaev (2007) [8] and David (2008) [21] adopt structural models including regime switching to assess the influence of different states of the economic cycles on the credit-risk premia.

In our framework, the state variables follow discrete-time conditionally Gaussian processes.<sup>2</sup> Extending the work of Gourieroux, Monfort and Polimenis (2006) [38], the Gaussian processes present drifts and variance-covariance matrices that are subject to regime shifts. The latter are described by a Markov chain with (historical) non-homogenous transition probabilities. Particular attention is paid to the tractability of the model and its estimation. Tractability is notably obtained through an extensive use of Car's –Compound autoregressive processes– properties (see, e.g. Darolles, Gourieroux and Jasiak, 2006 [20]), which leads to quasi-explicit formulas for bond prices. Both historical and risk-neutral dynamics are explicitly modeled, which is helpful for choosing appropriate specifications under the historical measure, for dealing simultaneously with pricing and forecasting, for Value-

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<sup>1</sup>Cathcart and El-Jahel, 2006 [13] have shown that the two approaches (reduced-form and structural) are somewhat reconcilable.

<sup>2</sup>While most of the earliest affine defaultable-bond term-structure models are in continuous-time form (see e.g. Duffie and Singleton (1999) [28]), Gourieroux, Monfort and Polimenis (2006) [38] have shown that discrete-time affine models are well-suited to credit-risk modeling and that they present higher flexibility than their continuous-time counterparts. In particular, the discrete-time framework makes it easier to properly specify the dynamics of the observable risk factors under the historical probability measure.

at-Risk calculations or for Sharpe-ratio computations.<sup>3</sup> We propose a sequential estimation strategy, which is intended to facilitate the estimation of unobservable factors (including latent risk factors and regimes).

The modeling of defaults is based on the so-called “doubly-stochastic” assumption: correlations between default events arise solely through dependence on some common underlying stochastic factors –also termed with “risk factors”– which influence the default probabilities of every single loans.<sup>4</sup> Some of the factors may be unobserved. In this sense, our model accomodates frailty. This feature is advocated by recent papers suggesting that including only observable covariates in default-intensity specifications results in poorly-estimated conditional probabilities of default (see e.g. Lando and Nielsen, 2008 [51] or Duffie et al., 2009 [27]).

Including regime shifts in a discrete-time term-structure model may affect pricing through several channels: (i) regimes affect the historical and risk-neutral dynamics of the risk factors, (ii) regimes appear in the stochastic discount factor (s.d.f.) –which implies that regime-transition risk is priced– and (iii) regimes appear in the default-intensity functions. This results in a large degree of flexibility in the model specifications, which is illustrated by several numerical examples in the paper. In particular, since default intensities can be affected by the regime variable, our model is appropriate to capture default clustering. Indeed, if one regime implies dramatic increases in the default probabilities of all or part of the debtors, numerous defaults will simultaneously take place during this regime.

Beyond the enrichment of the specifications of the risk factors and those of the default intensities by introducing nonlinearities, the regime-switching feature can be exploited to handle specific forms of contagions. Contagion effects, whose consequences are cascades of subsequent spread changes, is explained by the existence of close ties between firms (see, e.g., Jarrow and Yu, 2001 [47], Davies and Lo, 2001 [24] or Giesecke, 2004 [36]). Contagion takes place when the default probability of any debtor can be affected by the default event of another one. Given that our baseline model relies on the doubly-stochastic or conditional-dependence assumption –which states that, conditional to the underlying factors and regimes, the default events of the firms in a portfolio are independent– direct contagion effects can not be captured. Nevertheless, we can model specific contagion effects in two distinct ways. First, our framework can accomodate the specific contagion case where one entity –or, for the sake of tractability, a small number of them– affects the default probability of the others: it suffices to make one of the regimes corresponds to the default state of this entity. Second, the regime-switching feature can be exploited in order to capture “sector-contagion” phenomena. The sectors can represent different industries or different geographical areas. Each sector can be “infected” or not. When a sector gets infected, the default intensities of its constituents (the debtors) shift upwards. In this context, sector contagion stems from the parameterizations of the matrix of regime-transition probabilities. For instance, you can easily model infection probabilities that depend positively on the number of sectors already infected.

Our baseline model considers only one credit event: the default of the debtor. However, credit events include more generally the changes in credit ratings like these attributed by agencies like Moody’s, Standard & Poor’s or Fitch. There are several reasons why it may be desirable to model not only default events but also rating transitions (see Cantor, 2004 [12] or Gagliardini and Gouriéroux, 2001 [35]).<sup>5</sup> It turns out that our framework can be adapted

<sup>3</sup>Regarding the latter point, see Duffee (2010) [26].

<sup>4</sup>In our framework, these shocks include both Gaussian shocks and regime-shift shocks.

<sup>5</sup>Several of the main credit models currently being used in the industry, such as J.P. Morgan’s CreditMetrics (1997) [48], draw on the credit-migration approach. For presentation, comparison and evaluation of these

to accomodate time-varying credit-rating migrations along the lines of Lando (1998) [50] while keeping quasi-explicit bond-pricing formulas.<sup>6</sup>

The remainder of the paper is organized as follows. Sections 2 and 3 respectively present the historical and risk-neutral dynamics of the variables. Section 4 gives the bond-pricing formulas with zero or non-zero recovery rates. Section 4 also provides numerical examples. Section 5 deals with internal-consistency restrictions that arise when yields or asset returns are included amongst the risk factors. In Section 6, we propose an estimation strategy. Section 7 shows how the model accomodates the pricing of liquidity. Section 8 investigates possible extensions of the framework: Subsection 8.1 deals with multi-lag dynamics of the risk factors; Subsection 8.2 deals with the specific case where one of the Markov chains coincides with the default state of a given entity and Subsection 8.4 shows how to introduce rating-migration modeling in the framework. Section 9 concludes.

## 2. Information and historical dynamics

### 2.1. Information

The new information of the investors at date  $t$  is  $w'_t = (z'_t, y'_t, x'_t, d'_t)$  where  $z_t$  is a regime variable that can take a finite number  $J$  of values,  $y_t$  is a multivariate macroeconomic factor,  $x'_t = (x'_{1,t}, \dots, x'_{N,t})$  is a set of specific multivariate factors  $x_{n,t}$  associated with debtor  $n$ , and  $d'_t = (d_{1,t}, \dots, d_{N,t})$  is a set of binary variables indicating the default ( $d_{n,t} = 1$ ) or the non-default ( $d_{n,t} = 0$ ) state of entity  $n$ . The whole information set of the investors at date  $t$  is  $\underline{w}'_t = (w'_1, \dots, w'_t)$ . At this stage, we do not make any assumption about the observability of these variables by the econometrician (this is done below in Section 6). These regimes influence bond pricing through different channels (they will appear in the dynamics of the risk factors  $y_t$  and  $x_{n,t}$ 's, in the stochastic discount factor and in the default-intensity functions). In the baseline framework, the regimes are viewed as transitory: none of these regimes is absorbing but this restriction is relaxed in a specific case presented in Subsection 8.2.

### 2.2. Historical dynamics

It is convenient to make the regime variable  $z_t$  valued in  $\{e_1, \dots, e_J\}$ , the set of column vectors of the identity matrix  $I_J$ .<sup>7</sup> The conditional distribution of  $z_t$  given  $\underline{w}_{t-1}$  is characterized by the probabilities:

$$p(z_t | \underline{w}_{t-1}) = \pi(z_t | z_{t-1}, y_{t-1}). \quad (1)$$

The probability  $\pi(e_j | e_i, y_{t-1})$  that  $z_t$  shifts from regime  $i$  to regime  $j$  between period  $t-1$  and  $t$ , conditional on  $y_{t-1}$ , is also denoted by  $\pi_{ij,t-1}$ . These specifications allow for state-dependent transition probabilities, as in Gray (1996) [39], Ang and Bekaert (2002) [2] or Dai, Singleton and Yang (2007) [19].

The conditional distribution of  $y_t$  given  $z_t$  and  $\underline{w}_{t-1}$  is Gaussian and given by:

$$y_t = \mu(z_t, z_{t-1}) + \Phi y_{t-1} + \Omega(z_t, z_{t-1}) \varepsilon_t \quad (2)$$

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models, see e.g. Gordy (2000) [37].

<sup>6</sup>Other examples of term-structure models allowing for time-varying rating-migration probabilities include Bielecki and Rutkowski (2000) [9] and Wei (2003) [61].

<sup>7</sup>Indeed, this implies that any function of the regimes taking the value  $f_j$  in the  $j^{th}$  regime, say, is the linear function of  $z_t$ :  $f'z_t$  with  $f' = (f_1 \dots f_J)$ .

where the  $\varepsilon_t$  are independently and identically  $N(0, I)$  distributed. Specifications (1) and (2) imply that, in the universe  $(z_t, y_t)$ ,  $z_t$  Granger-causes  $y_t$ ,  $y_t$  causes  $z_t$  and there is instantaneous causality between  $z_t$  and  $y_t$ . Moreover, in the universe  $w_t = (z_t, y_t, x_t, d_t)$ ,  $(x_t, d_t)$  does not cause  $(z_t, y_t)$ . As noted by Ang, Bekaert and Wei (2008) [3], instantaneous causality between  $z_t$  and  $y_t$  implies that the variances of the factors  $y_t$ , conditional on  $\underline{w}_{t-1}$ , embed a jump term reflecting the difference in drifts  $\mu$  accross regimes. Such a feature, that allows for conditional heteroskedasticity, is absent from the Dai, Singleton and Yang (2007) [19] setting. However, it should be noted that our framework nests the case where there is no instantaneous causality between  $z_t$  and  $y_t$  in the historical dynamics.<sup>8</sup> Contrary to Bansal and Zhou (2002) [5], matrix  $\Phi$  is not regime-dependent: this is for the sake of tractability when it comes to bond pricing.<sup>9</sup>

The  $x_{n,t}$ 's,  $n = 1, \dots, N$  are assumed to be independent conditionally to  $(z_t, y_t, \underline{w}_{t-1})$ . The conditional distribution of  $x_{n,t}$  is Gaussian and defined by:

$$x_{n,t} = q_{1n}(z_t, z_{t-1}) + Q_{2n}y_t + Q_{3n}y_{t-1} + Q_{4n}x_{n,t-1} + Q_{5n}(z_t, z_{t-1})\eta_{n,t} \quad (3)$$

where the shocks  $\eta_{n,t}$  are  $IIN(0, I)$ . Specifications (1), (2) and (3) imply that, in the universe  $(z_t, y_t, x_{n,t})$ ,  $(z_t, y_t)$  causes  $x_{n,t}$ ,  $x_{n,t}$  does not cause  $(z_t, y_t)$  and there is instantaneous causality between  $(z_t, y_t)$  and  $x_{n,t}$ . Moreover, denoting by  $\bar{x}_{n,t}$  the vector  $x_t$  excluding  $x_{n,t}$ ,  $(\bar{x}_{n,t}, d_t)$  does not cause  $(z_t, y_t, x_{n,t})$  in the whole universe  $w_t$ .

Finally, the  $d_{n,t}$ 's,  $n = 1, \dots, N$ , are independent conditionally to  $(z_t, y_t, x_t, \underline{w}_{t-1})$  and the conditional distribution of  $d_{n,t}$  is such that:

$$p(d_{n,t} = 1 \mid z_t, y_t, x_t, \underline{w}_{t-1}) = \begin{cases} 1 & \text{if } d_{n,t-1} = 1, \\ 1 - \exp(-\lambda_{n,t}) & \text{otherwise,} \end{cases} \quad (4)$$

with  $\lambda_{n,t} = \alpha'_n z_t + \beta'_n y_t + \gamma'_n x_{n,t}$ .

In other words, state 1 of  $d_{n,t}$  is an absorbing state and  $\exp(-\lambda_{n,t})$  is the survival probability. Since the default probability  $1 - \exp(-\lambda_{n,t})$  is close to  $\lambda_{n,t}$  if  $\lambda_{n,t}$  is small,  $\lambda_{n,t}$  is called the default intensity. The default intensity is expected to be positive, which is not necessarily the case since the  $\varepsilon_t$ 's are Gaussian. However, the parameterization of the model may make this extremely unfrequent.

So, in the universe  $(z_t, y_t, x_{n,t}, d_{n,t})$ ,  $(z_t, y_t, x_{n,t})$  causes  $d_{n,t}$  whereas  $d_{n,t}$  does not causes  $(z_t, y_t, x_{n,t})$  and there is instantaneous causality. In the whole universe  $w_t$ ,  $(\bar{x}_{n,t}, \bar{d}_{n,t})$  does not cause  $(z_t, y_t, x_{n,t}, d_{n,t})$ . The causality scheme is summarized in Figure 1.

Finally, let us consider the conditional Laplace transform of the distribution of  $(z_t, y_t)$  given  $\underline{w}_{t-1}$ :

$$\varphi_{t-1}(u, v) = E_{t-1} [\exp(u'z_t + v'y_t)].$$

**Proposition 1.** *The conditional Laplace transform of  $(z_t, y_t)$  given  $\underline{w}_{t-1}$  is:*

$$\varphi_{t-1}(u, v) = \exp(v'\Phi y_{t-1} + [l_1, \dots, l_J] z_{t-1}), \quad (5)$$

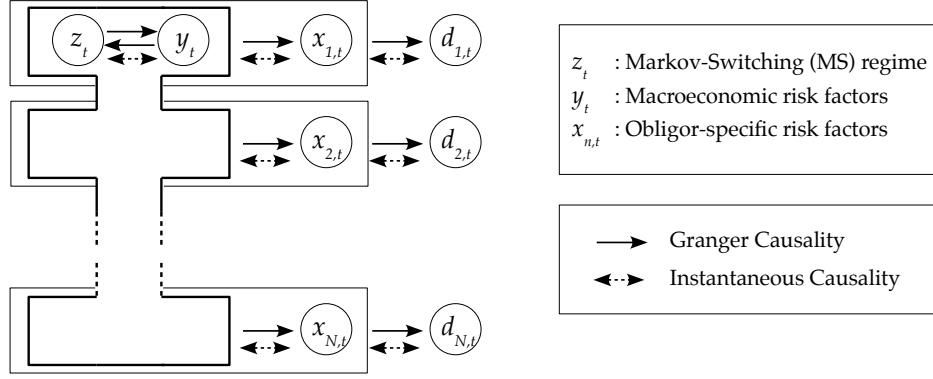
where  $l_i = \log \sum_{j=1}^J \pi_{ij,t-1} \exp\{u_j + v'\mu(e_j, e_i) + \frac{1}{2}v'\Omega(e_j, e_i)\Omega'(e_j, e_i)v\}$ .

<sup>8</sup>Formally, this corresponds to  $\mu(z_t, z_{t-1}) = \mu(z_{t-1})$  and  $\Omega(z_t, z_{t-1}) = \Omega(z_{t-1})$ .

<sup>9</sup>Indeed, the model of Bansal and Zhou (2002) [5] does not admit a closed-form exponential affine solution (they proceed by linearizing the discrete-time Euler equations and by solving the resulting linear relations for prices).



Figure 1: Causality scheme



*Proof.* We have

$$\begin{aligned}
 \varphi_{t-1}(u, v) &= E_{t-1}(\exp[u'z_t + v'y_t]) \\
 &= E_{t-1}(\exp[u'z_t + v'\mu(z_t, z_{t-1}) + v'\Phi y_{t-1} + v'\Omega(z_t, z_{t-1})\varepsilon_t]) \\
 &= E(E\{\exp[u'z_t + v'\mu(z_t, z_{t-1}) + v'\Phi y_{t-1} + v'\Omega(z_t, z_{t-1})\varepsilon_t] \mid \underline{w}_{t-1}, z_t\} \mid \underline{w}_{t-1}) \\
 &= \exp(v'\Phi y_{t-1}) E(\exp\{u'z_t + v'\mu(z_t, z_{t-1})\} \times E(\exp\{v'\Omega(z_t, z_{t-1})\varepsilon_t \mid \underline{w}_{t-1}, z_t\}) \mid \underline{w}_{t-1}) \\
 &= \exp(v'\Phi y_{t-1}) E(\exp\{u'z_t + v'\mu(z_t, z_{t-1})\} \times \frac{1}{2}v'\Omega(z_t, z_{t-1})\Omega'(z_t, z_{t-1})v \mid \underline{w}_{t-1}) \\
 &= \exp(v'\Phi y_{t-1} + [l_1, \dots, l_J]z_{t-1}).
 \end{aligned}$$

Using the expression given for the  $l_i$ 's leads to the result.  $\square$

This Laplace transform is not, in general, exponential affine in  $(z_{t-1}, y_{t-1})$ , since  $y_{t-1}$  appears in the  $\pi_{ij,t}$ 's. However, this is the case if the  $\pi_{ij,t}$ 's do not depend on  $y_{t-1}$  and then  $(z_t, y_t)$  is Car(1) (see Darolles, Gouriéroux and Jasiak, 2006[20] or Bertholon, Monfort and Pegoraro (2008) [7] for in-depth presentations of Car processes).

### 3. Stochastic discount factor and risk-neutral dynamics

#### 3.1. Stochastic discount factor

We complete the model by specifying the stochastic discount factor  $M_{t-1,t}$  between  $t-1$  and  $t$ :

$$\begin{aligned}
 M_{t-1,t} &= \exp\left[-a'_1 z_{t-1} - b'_1 y_{t-1} - \frac{1}{2}\nu'(z_t, z_{t-1}, y_{t-1})\nu(z_t, z_{t-1}, y_{t-1}) + \right. \\
 &\quad \left. + \nu'(z_t, z_{t-1}, y_{t-1})\varepsilon_t + \delta'(z_{t-1}, y_{t-1})z_t\right],
 \end{aligned} \tag{6}$$

with the constraints:

$$\sum_{j=1}^J \pi_{ij,t-1} \exp[\delta_j(e_i, y_{t-1})] = 1, \forall i, y_{t-1}, \tag{7}$$

where  $\delta_j$  is the  $j^{th}$  component of  $\delta$ . Using Equation (7), it is easily seen that  $E_{t-1}(M_{t-1,t}) = \exp(-a_1' z_{t-1} - b_1' y_{t-1})$ . Therefore, the riskless short rate between  $t-1$  and  $t$  is:

$$r_t = a_1' z_{t-1} + b_1' y_{t-1}. \quad (8)$$

In our framework, the variables  $(x_{n,t}, d_{n,t})$ , specific to entity  $n$ , do not appear in the stochastic discount factor. This means that these entities have no impact at the macroeconomic level.<sup>10</sup> This can be formalised in the following way. Let us assume that the  $N$  entities appearing in the modeling belong to a large population of size  $\tilde{N}$ . This large population could appear in  $M_{t-1,t}$ , for instance through a term of the form

$$G_t(\tilde{N}) = \frac{1}{\tilde{N}} \sum_{n=1}^{\tilde{N}} \left( \nu_n' x_{n,t} + \nu_{0n}' d_{n,t} \right).$$

Since the  $(x_{n,t}, d_{n,t})$ ,  $i = 1, \dots, \tilde{N}$  are independent conditionally to  $\underline{z}_t, \underline{y}_t$ , we have, denoting respectively by  $E_t$  and  $V_t$  the conditional expectation and variance (or variance-covariance matrix) given  $\underline{z}_t, \underline{y}_t$ :

$$V_t \left( G_t(\tilde{N}) \right) = \frac{1}{\tilde{N}^2} \sum_{n=1}^{\tilde{N}} \left[ \nu_n', \nu_{0n} \right] V_t \left( x_{n,t}', d_{n,t} \right) \left[ \nu_n', \nu_{0n} \right]'$$

Assuming that the terms in the sum are bounded when  $\tilde{N}$  goes to infinity, which means that all the entities have a bounded weight in the infinite population,  $V_t(G_t(\tilde{N}))$  goes to zero, when  $\tilde{N}$  goes to infinity and  $G_t(\tilde{N})$  converges in mean square to  $\lim_{\tilde{N} \rightarrow \infty} E_t(G_t(\tilde{N}))$  (which is assumed to exist). Therefore,  $G_t(\tilde{N})$  asymptotically depends only on  $(\underline{z}_t, \underline{y}_t)$ , which already appears in  $M_{t-1,t}$ . In some sense, the impact of these entities has been diversified away.

So the framework of this paper can be used in the context described above, the entities appearing in the modeling are those of specific interest, and the sequential inference method proposed in section 6 shows that these entities can be incorporated progressively in the model.

## 3.2. Risk-neutral dynamics

### 3.2.1. The conditional risk-neutral distribution of $(z_t, y_t)$ given $\underline{w}_{t-1}$

Let us now consider the conditional risk-neutral Laplace transform of  $(z_t, y_t)$  given  $\underline{w}_{t-1}$ ,  $\varphi_{t-1}^{\mathbb{Q}}(u, v) := E_{t-1}^{\mathbb{Q}}(\exp[u' z_t + v' y_t])$ , and let us introduce the simplified notations:

$$\begin{aligned} \mu_t &= \mu(z_t, z_{t-1}) \\ \Omega_t &= \Omega(z_t, z_{t-1}), \Sigma(z_t, z_{t-1}) = \Omega_t \Omega_t' = \Sigma_t \\ \nu_t &= \nu(z_t, z_{t-1}, y_{t-1}) \\ \delta_{t-1} &= \delta(z_{t-1}, y_{t-1}). \end{aligned}$$

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<sup>10</sup>Diversifiability assumptions and the implied restrictions on default risk premia are studied in details by Jarrow, Lando and Yu (2005) [46] (in a continuous-time setting).

**Proposition 2.** *The conditional risk-neutral Laplace transform of  $(z_t, y_t)$  given  $\underline{w}_{t-1}$  is:*

$$\varphi_{t-1}^{\mathbb{Q}}(u, v) = \exp \left[ v' \Phi y_{t-1} + \begin{pmatrix} A_{1,t-1}(u, v) & \dots & A_{J,t-1}(u, v) \end{pmatrix} z_{t-1} \right], \quad (9)$$

where

$$A_{i,t-1}(u, v) = \log \left( \sum_{j=1}^J \pi_{ij,t-1} \exp \left\{ v' \Omega(e_j, e_i) \nu(e_j, e_i, y_{t-1}) + \frac{1}{2} v' \Sigma(e_j, e_i) v + v' \mu(e_j, e_i) + u_j + \delta_j(e_i, y_{t-1}) \right\} \right).$$

*Proof.* See Appendix A.1. □

We immediately deduce the following Corollary.

**Corollary 1.** *The risk-neutral dynamics of  $(z_t, y_t)$  is  $Car(1)$  if the s.d.f. satisfies the constraints (for any  $i, j$  and  $t$ ):*

$$\begin{cases} \pi(e_j | e_i, y_{t-1}) \exp[\delta_j(e_i, y_{t-1})] &= \pi_{ij}^* \\ \Omega(e_j, e_i) \nu(e_j, e_i, y_{t-1}) &= \Phi^* y_{t-1} + \mu^*(e_j, e_i), \end{cases} \quad (10)$$

where  $\pi_{ij}^* = \pi^*(e_j | e_i)$  does not depend on  $y_{t-1}$ ,  $\Phi^*$  is any matrix and  $\mu^*$  is any function.

If such constraints are satisfied, the risk-neutral conditional Laplace transform becomes:

$$\varphi_{t-1}^{\mathbb{Q}}(u, v) = \exp \left[ v' (\Phi + \Phi^*) y_{t-1} + \begin{pmatrix} A_1^*(u, v) & \dots & A_J^*(u, v) \end{pmatrix} z_{t-1} \right], \quad (11)$$

with  $A_i^*(u, v) = \log \left( \sum_{j=1}^J \pi_{ij}^* \exp \left\{ u_j + v' [\mu(e_j, e_i) + \mu^*(e_j, e_i)] + \frac{1}{2} v' \Sigma(e_j, e_i) v \right\} \right).$

Comparing with equation (5), we deduce that the risk-neutral dynamics of  $(z_t, y_t)$  is then defined by:

$$y_t = \mu(z_t, z_{t-1}) + \mu^*(z_t, z_{t-1}) + (\Phi + \Phi^*) y_{t-1} + \Omega(z_t, z_{t-1}) \varepsilon_t^*, \quad (12)$$

where, under  $\mathbb{Q}$ ,  $z_t$  is an homogenous Markov chain defined by the transition matrix  $\{\pi_{ij}^*\}$ , and  $\varepsilon_t^*$ —defined by  $\varepsilon_t^* = \varepsilon_t - \Omega^{-1}(z_t, z_{t-1}) [\mu^*(z_t, z_{t-1}) + \Phi^* y_{t-1}]$ —is  $IIN(0, I)$ .

The previous results show that an appropriate choice of the s.d.f., that is an appropriate choice of the risk sensitivity vectors  $\nu$  and  $\delta$  pricing respectively the (standardized) innovations  $\varepsilon_t$  of  $y_t$  and the regimes  $z_t$ , allows to obtain a joint risk-neutral dynamics of  $(z_t, y_t)$  defined by any transition matrix  $\{\pi_{ij}^*\}$  and any equation:

$$y_t = \tilde{\mu}(z_t, z_{t-1}) + \tilde{\Phi} y_{t-1} + \Omega(z_t, z_{t-1}) \varepsilon_t^*,$$

where  $\varepsilon_t^*$  is  $IIN(0, I)$ . Note that the  $\Omega$  function is the same in the historical and risk-neutral worlds.

### 3.2.2. The risk-neutral distribution of $(x_t, d_t)$ given $(z_t, y_t, \underline{w}_{t-1})$

**Lemma 1.** *Let us consider a partition of  $w_t = (w'_{1,t}, w'_{2,t})'$ . If  $M_{t-1,t}$  is a function of  $(w_{1,t}, \underline{w}_{t-1})$ , the risk-neutral probability density function, or p.d.f., of  $w_{1,t}$  given  $\underline{w}_{t-1}$  is:*

$$f_{\mathbb{Q}}(w_{1,t} | \underline{w}_{t-1}) = f(w_{1,t} | \underline{w}_{t-1}) M_{t-1,t} \exp(-r_t)$$

(where  $f$  is the historical conditional p.d.f. of  $w_{1,t}$  given  $\underline{w}_{t-1}$ ) and the conditional risk-neutral distribution of  $w_{2,t}$  given  $(w_{1,t}, \underline{w}_{t-1})$  is the same as the corresponding historical distribution.

*Proof.* See Appendix A.2. □

Since  $M_{t-1,t}$  is a function of  $(z_t, y_t)$  but not of  $(x_t, d_t)$ , the previous lemma shows that the risk-neutral distribution of  $(x_t, d_t)$  given  $(z_t, y_t, \underline{w}_{t-1})$  is the same as the historical one and it is given by equations (3) and (4). In particular, the functional forms of the default intensities  $\lambda_{n,t}$  are the same as in the historical world. Of course, since the dynamics of  $(z_t, y_t)$  are different in the two worlds, the same is true for the  $x_{n,t}$ 's and the  $\lambda_{n,t}$ 's.

In addition, it can be shown that  $(z_t, y_t, x_{n,t})$  is Car(1) under the risk-neutral measure (see Appendix A.3). However, it is not the case for  $(z_t, y_t, x_{n,t}, d_{n,t})$ .

It is also clear that the causality structure of the risk-neutral dynamics is similar to the historical one, the only difference being the non-causality from  $y_t$  to  $z_t$  implied by the homogeneity of the matrix  $\{\pi_{ij}^*\}$ .

### 3.3. Discussion of the constraints on the SDF

Constraints (10) can be written:

$$\begin{cases} \delta_j(z_{t-1}, y_{t-1}) &= \log \left( \frac{\pi^*(e_j | z_{t-1})}{\pi(e_j | z_{t-1}, y_{t-1})} \right) \\ \nu(z_t, z_{t-1}, y_{t-1}) &= \Omega^{-1}(z_t, z_{t-1}) [\Phi^* y_{t-1} + \mu^*(z_t, z_{t-1})], \end{cases} \quad (13)$$

where the transition matrix  $\{\pi_{ij}^*\}$ , the matrix  $\Phi^*$  and the vectors  $\mu^*(e_j, e_i)$  are arbitrary. Note that constraints (7) imposed on  $\delta$  are automatically satisfied by the parameterization (13). Recall that constraints (10) are imposed so as to obtain a Car dynamics of the state variable under the risk-neutral measure. These constraints could be relaxed, but at the cost of losing the analytical tractability in the bond pricing (as will be shown below). Even if we impose a Car risk-neutral dynamics, we still have a large number of degrees of freedom in the specification of the s.d.f. since  $\Phi^*$ ,  $\mu^*(z_t, z_{t-1})$  and the  $\pi_{ij}^*$ 's are then chosen arbitrarily. However, we may wish to parameterize more parsimoniously the s.d.f. and, therefore, impose stronger constraints on the risk-neutral dynamics. Let us illustrate this point by a simple bivariate example.

The historical dynamics is defined by:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu'_2 z_t \end{bmatrix} + \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \sigma_1 \varepsilon_{1,t} \\ (\sigma'_2 z_t) \varepsilon_{2,t} \end{bmatrix}$$

and by some  $\pi_{ij,t}$ 's. Moreover, let us assume that we impose an additive risk-sensitivity vector  $\nu$ :

$$\nu(z_t, z_{t-1}, y_{t-1}) = \begin{pmatrix} b'_1 y_{t-1} + \nu'_1 z_t \\ b'_2 y_{t-1} + \nu'_2 z_t \end{pmatrix}.$$

We get:

$$\Omega(z_t, z_{t-1})\nu(z_t, z_{t-1}, y_{t-1}) = \begin{bmatrix} \sigma_1 b'_1 y_{t-1} + \sigma_1 \nu'_1 z_t \\ \sigma'_2 z_t (b'_2 y_{t-1} + \nu'_2 z_t) \end{bmatrix},$$

which must be additive of the form  $\Phi^* y_{t-1} + \mu^*(z_t, z_{t-1})$ . It is only possible if  $b_2 = 0$  and in this case we get:

$$\Phi^* = \begin{bmatrix} \sigma_1 b'_1 \\ 0 \end{bmatrix} \text{ and } \mu^*(z_t, z_{t-1}) = \begin{bmatrix} \sigma_1 \nu'_1 z_t \\ (\sigma_2 \odot \nu_2)' z_t \end{bmatrix},$$

where  $\odot$  denotes the Hadamard (element by element) product. In other words  $\Phi^* = \begin{bmatrix} \varphi_1^* \\ 0 \end{bmatrix}$ ,  $\mu^*(z_t, z_{t-1}) = \begin{bmatrix} \mu_1^* z_t \\ \mu_2^* z_t \end{bmatrix}$  where  $\varphi_1^*$ ,  $\mu_1^*$  and  $\mu_2^*$  are arbitrary. Finally, the risk-neutral dynamics is given by:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} \tilde{\mu}'_1 z_t \\ \tilde{\mu}'_2 z_t \end{bmatrix} + \begin{bmatrix} \tilde{\varphi}_{11} & \tilde{\varphi}_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \sigma_1 \varepsilon_{1,t}^* \\ (\sigma'_2 z_t) \varepsilon_{2,t}^* \end{bmatrix}$$

and by  $\{\pi_{ij}^*\}$  where  $\tilde{\varphi}_{11}$ ,  $\tilde{\varphi}_{12}$ ,  $\tilde{\mu}_1$ ,  $\tilde{\mu}_2$  and the  $\pi_{ij}^*$ 's are arbitrary, but the autoregressive coefficients of the second equations are the same as in the historical dynamics.

## 4. Pricing

### 4.1. Pricing of riskless zero-coupon bonds

It is well-known that the existence of a positive stochastic discount factor is equivalent to the absence of arbitrage opportunities (see Hansen and Richard, 1987 [43] and Berholon, Monfort and Pegoraro, 2007 [7]) and that the price at  $t$  of a zero-coupon bond with residual maturity  $h$  is given by:

$$B(t, h) = E_t^{\mathbb{Q}} [\exp(-r_{t+1} - \dots - r_{t+h})], \quad (14)$$

where  $r_{t+i} = a'_1 z_{t+i-1} + b'_1 y_{t+i-1}$ ,  $i = 1, \dots, h$ . Since  $(z_t, y_t)$  is Car(1) under  $\mathbb{Q}$ ,  $B(t, h)$  is easily computed using the following lemma:

**Lemma 2.** *Let us consider a multivariate Car(1) process  $Z_t$  and its conditional Laplace transform given by  $\exp[a'(s)Z_t + b(s)]$ . Let us further denote by  $L_{t,h}(\omega)$  its multi-horizon Laplace transform given by:*

$$L_{t,h}(\omega) = E_t [\exp(\omega'_{H-h+1} Z_{t+1} + \dots + \omega'_H Z_{t+h})], \quad t = 1, \dots, T, \quad h = 1, \dots, H,$$

where  $\omega = (\omega'_1, \dots, \omega'_H)$  is a given sequence of vectors. We have, for any  $t$ ,

$$L_{t,h}(\omega) = \exp(A'_h Z_t + B_h), \quad h = 1, \dots, H,$$

where the sequences  $A_h$ ,  $B_h$ ,  $h = 1, \dots, H$  are obtained recursively by:

$$\begin{aligned} A_h &= a(\omega_{H-h+1} + A_{h-1}) \\ B_h &= b(\omega_{H-h+1} + A_{h-1}) + B_{h-1}, \end{aligned}$$

with the initial conditions  $A_0 = 0$  and  $B_0 = 0$ .

*Proof.* See Appendix A.4. □

From Equation (11) we know that  $(z_t, y_t)$  is risk-neutral  $\text{Car}(1)$  and that its conditional Laplace transform is based on the functions:

$$\begin{aligned} a'(u, v) &= [(A_1^*(u, v), \dots, A_J^*(u, v)), v'(\Phi + \Phi^*)] \text{ and} \\ b(u, v) &= 0. \end{aligned}$$

so we have the following proposition:

**Proposition 3.** *We have:*

$$B(t, h) = \exp \left( -a'_h z_t - b'_h y_t \right), \quad (15)$$

and the yield of residual maturity  $h$ ,  $R(t, h)$  is given by:

$$R(t, h) = \frac{1}{h} \left( a'_h z_t + b'_h y_t \right), \quad (16)$$

where  $a_h$  and  $b_h$  are computed recursively, for  $h = 1, \dots, H$ , by (with  $a_0 = a_1$  and  $b_0 = b_1$ ):

$$(a'_h, b'_h) = (a'_1, b'_1) - a' \left( \omega_{H-h+1} - (a'_{h-1} - a'_1, b'_{h-1} - b'_1)' \right),$$

where the sequence  $\omega_h$ ,  $h = 1, \dots, H$  is defined by  $\omega_H = 0$ ,  $\omega_1 = \omega_2 = \dots = \omega_{H-1} = (-a'_1, -b'_1)'$  and where  $a'(u, v) = [(A_1^*(u, v), \dots, A_J^*(u, v)), v'(\Phi + \Phi^*)]$ .

*Proof.* See Appendix A.5. □

## 4.2. Pricing of (zero-recovery-rate) defaultable bonds

A defaultable zero-coupon bond providing one money unit at  $t + h$  if entity  $n$  is still alive at  $t + h$  and zero otherwise has a price at  $t$  given by:

$$B_n^D(t, h) = E_t^{\mathbb{Q}} \left[ \exp(-r_{t+1} - \dots - r_{t+h}) \mathbb{I}_{\{d_{n,t+h}=0\}} \right] \quad (17)$$

if  $d_{n,t} = 0$  and 0 otherwise.

**Proposition 4.** *The price of a zero-recovery-rate zero-coupon defaultable bond issued by debtor  $n$  is such that:*

$$\begin{aligned} B_n^D(t, h) &= E_t^{\mathbb{Q}} \left[ \exp \left( -r_{t+1} - \dots - r_{t+h} - \alpha'_n z_{t+1} - \beta'_n y_{t+1} - \gamma'_n x_{n,t+1} - \dots \right. \right. \\ &\quad \left. \left. - \dots - \alpha'_n z_{t+h} - \beta'_n y_{t+h} - \gamma'_n x_{n,t+h} \right) \right]. \end{aligned} \quad (18)$$

*Proof.* Equation (17) can be rewritten:

$$\begin{aligned} B_n^D(t, h) &= E_t^{\mathbb{Q}} \left[ E^{\mathbb{Q}} \left( \exp(-r_{t+1} - \dots - r_{t+h}) \mathbb{I}_{\{d_{n,t+h}=0\}} \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, d_{n,t} = 0 \right) \right] \\ &= E_t^{\mathbb{Q}} \left[ \exp(-r_{t+1} - \dots - r_{t+h}) \mathbb{Q} \left( d_{n,t+h} = 0 \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, d_{n,t} = 0 \right) \right]. \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathbb{Q} \left( d_{n,t+h} = 0 \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, d_{n,t} = 0 \right) \\ &= \prod_{i=1}^h \mathbb{Q} \left( d_{n,t+i} = 0 \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, d_{n,t+i-1} = 0 \right) \end{aligned}$$

and, since  $d_{n,t}$  does not  $\mathbb{Q}$ -cause  $(z_t, y_t, x_{n,t})$  in the Granger's or Sims' sense, we have:<sup>11</sup>

$$\begin{aligned} &\mathbb{Q} \left( d_{n,t+i} = 0 \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, d_{n,t+i-1} = 0 \right) \\ &= \mathbb{Q} \left( d_{n,t+i} = 0 \mid \underline{z}_{t+i}, \underline{y}_{t+i}, \underline{x}_{n,t+i}, d_{n,t+i-1} = 0 \right) \\ &= \exp(-\lambda_{n,t+i}). \end{aligned}$$

where the last equality comes from the fact that the conditional historical and risk-neutral distributions of  $d_{n,t}$  are the same (see Subsection 3.2.2).  $\square$

It can be shown (see Appendix A.3) that  $(z_t, y_t, x_{n,t})$  is Car(1) under  $\mathbb{Q}$ , with a conditional Laplace transform of the type  $\exp[\bar{a}'(u, v, w)(z'_t, y'_t, x'_{n,t})]$  where  $\bar{a}(u, v, w) = [(\tilde{A}_1, \dots, \tilde{A}_J), (v' + w'Q_{2n})(\Phi + \Phi^*) + w'Q_{3n}, w'Q_{4n}]$ , where

$$\begin{aligned} \tilde{A}_i(u, v, w) &= \log \left( \sum_{j=1}^J \pi_{ij}^* \exp \{ u_j + (v' + w'Q_{2n}) [\mu(e_j, e_i) + \mu^*(e_j, e_i)] + w'q_{1n}(e_j, e_i) + \right. \\ &\quad \left. \frac{1}{2}(v' + w'Q_{2n})\Sigma(e_j, e_i)(v + Q'_{2n}w) + \frac{1}{2}w'Q_{5n}(e_j, e_i)Q'_{5n}(e_j, e_i)w \} \right). \end{aligned}$$

Therefore we have the following result:

**Proposition 5.** *The price of a zero-recovery-rate zero-coupon defaultable bond issued by debtor  $n$  is given by:*

$$B_n^D(t, h) = \exp \left( -c'_{n,h}z_t - f'_{n,h}y_t - g'_{n,h}x_{n,t} \right) \quad (19)$$

and the defaultable yields are:

$$R_n^D(t, h) = \frac{1}{h} \left( c'_{n,h}z_t + f'_{n,h}y_t + g'_{n,h}x_{n,t} \right), \quad (20)$$

where  $(c'_{n,h}, f'_{n,h}, g'_{n,h})$  is computed recursively by:

$$(c'_{n,h}, f'_{n,h}, g'_{n,h}) = (a'_1, b'_1, 0) - \bar{a} \left( \omega_{H-h+1} - (c'_{n,h-1} - a'_1, f'_{n,h-1} - b'_1, -g'_{n,h-1})' \right)$$

where the sequence  $\omega_h$ ,  $h = 1, \dots, H$  is defined by  $\omega_H = (-\alpha'_n, -\beta'_n, -\gamma'_n)$  and  $\omega_h = (-\alpha'_n - a'_1, -\beta'_n - b'_1, -\gamma'_n)$  for  $h = 1, \dots, H-1$ , with  $c_{n,0} = a_1$ ,  $f_{n,0} = b_1$ ,  $g_{n,0} = 0$ .

<sup>11</sup>A process  $X_t$  does not cause  $Y_t$  in Granger's sense if and only if, for any  $t$ ,  $Y_t$  is independent of  $(X_{t-1}, \dots, X_1)$  conditionally on  $(Y_{t-1}, \dots, Y_1)$ . This is equivalent to the non-causality in Sims' sense ( $X_t$  does not cause the stochastic process  $Y_t$  in Sims' sense iff  $X_t$  is independent from  $(Y_{t+1}, Y_{t+2}, \dots, Y_T)$  conditionally on  $(Y_t, X_{t-1}, Y_{t-1}, \dots, X_1, Y_1)$ ).

*Proof.* See Appendix A.6. □

In this setting, credit spreads are given by:

$$\begin{aligned} s_n(t, h) &= R_n^D(t, h) - R_n(t, h) \\ &= \frac{1}{h} \left[ (c_{n,h} - a_h)' z_t + (f_{n,h} - b_h)' y_t + g_h' x_{n,t} \right]. \end{aligned} \quad (21)$$

Equations (15), (20) and (21) show that yields and spreads are given by some combinations of discrete and real-valued factors, the latter following Markov-Switching vector autoregressive models defined by equations (2) and (3). Therefore, yields and spreads present (conditional and unconditional) distributions that can be far richer than the Gaussian ones that would be obtained in the absence of regime-switching. This will be illustrated in a numerical example below (subsection 4.4).

### 4.3. Pricing of non-zero-recovery-rate defaultable bonds

Formula (18), which can read

$$B_n^D(t, h) = E_t^{\mathbb{Q}} [\exp(-r_{t+1} - \dots - r_{t+h} - \lambda_{n,t+1} - \dots - \lambda_{n,t+h})], \quad (22)$$

has been obtained under the assumption of zero recovery rate. This formula can be extended to the case with non-zero recovery rates, providing that the  $\lambda_{n,t}$ 's are interpreted as risk-neutral “recovery-adjusted” default intensities. More precisely, we have the following result (dropping the subscript  $n$  for the sake of clarity):

**Proposition 6.** *If, for any bond issued by debtor  $n$  before  $t$ , the recovery payoff –that is assumed to be paid at time  $t$  in case of default between  $t-1$  and  $t$  of debtor  $n$ – is equal to the product of a function  $\zeta_{n,t}$  of the information available at time  $t$  by the survival-contingent market value of the bond at  $t$ , the price at  $t$  of a bond with residual maturity  $h$  is:*

$$B_n^{DR}(t, h) = E_t^{\mathbb{Q}} \left[ \exp(-r_{t+h} - \dots - r_{t+h} - \tilde{\lambda}_{n,t+1} - \dots - \tilde{\lambda}_{n,t+h}) \right], \quad (23)$$

where  $\tilde{\lambda}_{n,s}$  is defined by (for any  $s$ ):

$$\exp(-\tilde{\lambda}_{n,s}) = \exp(-\lambda_{n,s}) + (1 - \exp(-\lambda_{n,s})) \zeta_{n,s}.$$

*Proof.* See Appendix A.7. □

The assumption of Proposition 6 is similar to the “Recovery of Market Value” assumption made by Duffie and Singleton (1999) [28] except that, in their discrete-time approach, they assume that  $\zeta_t$  is known at time  $t-1$ , and that conditionally to the information at  $t-1$ ,  $d_{n,t}$  is independent of the recovery payoff at  $t$ .



## 4.4. Numerical example

This subsection proposes a simple numerical experiment to illustrate some properties of the model.

Let us consider a model with two regimes. Whereas the first regime ( $z_t = [1, 0]'$ ) corresponds to “normal times”, the second ( $z_t = [0, 1]'$ ) is intended to capture crisis periods. Table 1 defines the dynamics of macroeconomic and regime variables. A first macroeconomic variable, denoted by  $y_{r,t}$ , follows an AR(1) with innovation  $\varepsilon_t \sim NIID(0, 1)$ . A second macroeconomic variables, denoted by  $y_{z,t}$ , is deterministic conditionally to  $z_t$ . This variable acts like a memory of past crisis periods. Intuitively, if it was infinitely persistent (i.e., if we had  $y_{z,t} = [0 \ 1] z_t + y_{z,t-1}$ ),  $y_{z,t}$  would count the number of past crisis periods. The crisis regime is far more persistent under the risk-neutral measure than under the historical one: under the former measure, the life expectancy of the crisis regime is 4 periods while it is of 100 periods under the latter measure. The short rate is given by  $r_{t+1} = 0.04 + y_{r,t}$ .

Table 1 – Dynamics under both measures

|                                 | Under $\mathbb{P}$   | Under $\mathbb{Q}$   |
|---------------------------------|--|--|
| Dynamics of<br>macro. variables | $y_{r,t} = 0.6y_{r,t-1} + \varepsilon_t$<br>$y_{z,t} = [0 \ 1] z_t + 0.8y_{z,t-1}$ | $y_{r,t} = 0.3 + 0.8y_{r,t-1} + \varepsilon_t$<br>$y_{z,t} = [0 \ 1] z_t + 0.9y_{z,t-1}$ |
| Transition proba.               | $\{\pi_{i,j}\} = \begin{bmatrix} 0.98 & 0.02 \\ 0.25 & 0.75 \end{bmatrix}$         | $\{\pi_{i,j}^*\} = \begin{bmatrix} 0.98 & 0.02 \\ 0.01 & 0.99 \end{bmatrix}$             |

Let us consider two firms. These firms are characterized by their respective default intensities  $\lambda_{1,t}$  and  $\lambda_{2,t}$ :

$$\begin{cases} \lambda_{1,t} &= \begin{bmatrix} 0.01 & 0.03 \end{bmatrix} z_t + \begin{bmatrix} 0.002 & 0.000 \end{bmatrix} \begin{bmatrix} y_{r,t} & y_{z,t} \end{bmatrix}' \\ \lambda_{2,t} &= \begin{bmatrix} 0.01 & 0.01 \end{bmatrix} z_t + \begin{bmatrix} 0.002 & 0.010 \end{bmatrix} \begin{bmatrix} y_{r,t} & y_{z,t} \end{bmatrix}' \end{cases}$$

Both types of firms are affected during crises but not in the same manner. Specifically, whereas the default intensity of firm 1 depends directly on  $z_t$ , the effect of the crisis on the default intensity of firm 2 works through  $y_{z,t}$ . The upper panel of Figure 2 displays a simulated sample of  $(z_t, y_{r,t}, y_{z,t})$ . The lower panel shows the implied default intensities. It turns out that the default intensity of firm 2 reacts in a more progressive but also in a more persistent manner to crisis events than the default intensity of firm 1.

Figure 2: **Simulation of default intensities for the two types of firms**

Notes: The figure displays simulated trajectories of the macroeconomic factors  $y_{r,t}$  and  $y_{z,t}$  (upper panel) and of the associated default intensities for the two types of firms (lower panel). The grey-shaded areas indicate crisis periods (second regime  $z_t = \begin{bmatrix} 0 & 1 \end{bmatrix}$ ).

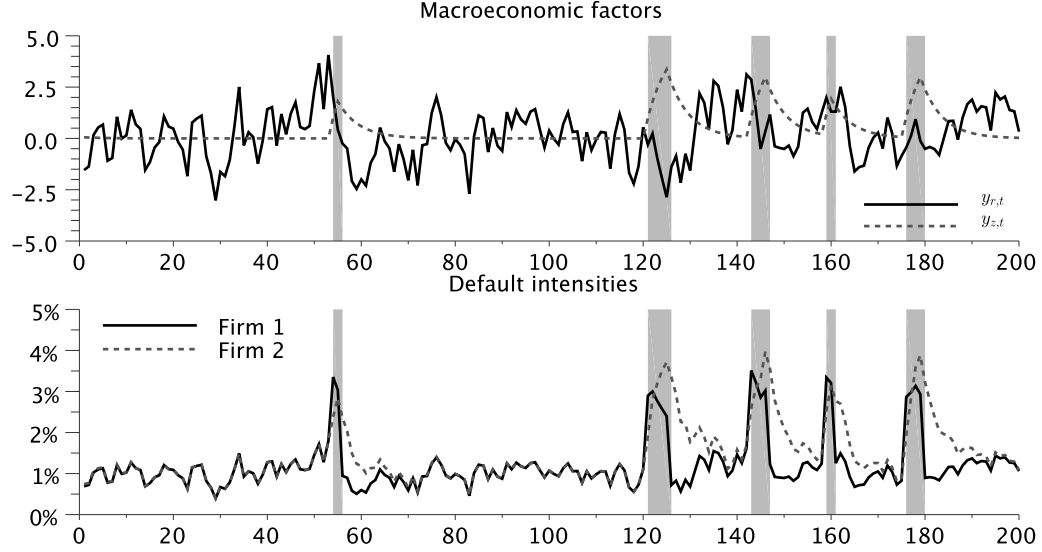
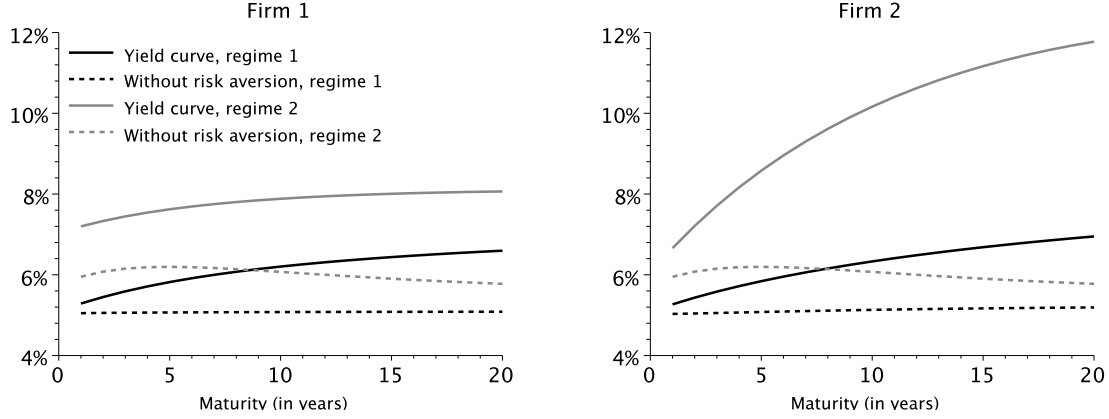


Figure 3 shows the yield curves for both firms when  $y_{r,t}$  is equal to its unconditional mean (zero) and when  $y_{z,t-1} = 0$ . As expected, for all maturities, yields are higher in the crisis regime (grey solid lines) than in the normal-times regime (black solid lines). The plots also report the yield curves that would be observed if the representative investor were risk-neutral or, equivalently, if the risk-neutral measure were identical to the historical one (dashed lines). The yield differentials between solid lines and dashed lines correspond to risk premia. These premia are particularly high in the crisis regime and for firm 2. Indeed, recall that the crisis regime is very persistent under the risk-neutral measure. Accordingly, there is a high risk-neutral probability –compared to the historical probability– that the crisis regime continues in the next periods. Since the default intensity of firm 2 depends on  $y_{z,t}$ , that itself depends on the number of crisis periods, the risk-neutral expectations of the default intensity increases with maturity. This generates the strong steepness of the yield curve for firm 2 under the crisis regime (grey solid line).

Figure 3: **Yield curves for both types of firms and under both regimes**

**Notes:** The solid lines represent yield curves observed when  $y_{r,t} = 0$  and  $y_{z,t-1} = 0$ . If the representative investor were risk neutral or, equivalently, if the risk-neutral dynamics were identical to the historical ones, the yield curves would be given by the dashed lines.



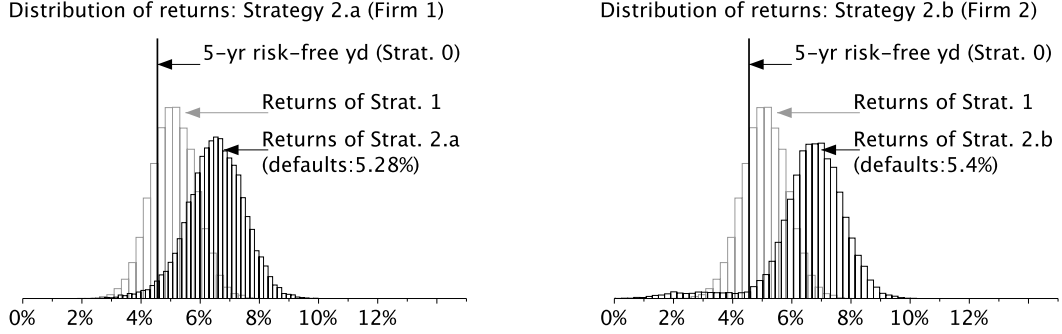
With such a model, one can perform many exercises, one of them being the computation the distribution of returns provided by various portfolios of defaultable and/or risk-free bonds. Let us consider an investor that wants to invest a given amount over, say, the next five years (between time  $t$  and time  $t + 5$ ). Assume that she wants to compare three specific strategies:

- Strategy 0: Purchase of 5-year risk-free zero-coupon bonds (hold till maturity),
- Strategy 1: In period  $t$ , purchase of 10-year risk-free zero-coupon bonds and sale of those bonds in  $t + 5$ ,
- Strategy 2.a and 2.b: Purchase of 10-year defaultable bonds (issued by firm 1 in Strategy 2.a and issued by firm 2 in Strategy 2.b) and sale of those bonds in  $t + 5$ .

Whereas the return provided by Strategy 0 is known ex ante (it is given by the 5-year risk-free yields), the returns associated with the other strategies are uncertain. As regards Strategy 1, uncertainty stems from the fact the price of a 5-year risk-free zero-coupon bond in  $t + 5$  years is not known at time  $t$ . For Strategies 2.a and 2.b, additional uncertainty comes from the possible default of the indebted firms. Figure 4 displays the distributions of the returns of these strategies. It turns out that, conditionally on the absence of default of firm 2, the return distribution is bi-modal: most of the lower returns (below 3%) correspond to drawings for which the crisis regime prevails at time  $t + 5$ , implying low prices for the bonds issued by firm 2 (as can be seen on Figure 3).

Figure 4: **(Annualized) Returns provided by investment Strategies 0, 1, 2.a and 2.b**

Notes: The return distributions are obtained by simulating 20.000 5-year scenrios. Initially,  $y_{r,t} = 0$ ,  $y_{z,t} = 0$  and  $z_t = [1, 0]'$ . The distribution of Strategy-0 returns is a Dirac mass at  $R(t, 5)$ . The distribution of Strategy-1 returns is represented in light grey (in black for those yielded by Strategy 2). Returns are logarithmic. Strategies-2 distributions do not include defaults events (but the proportion of defaults is reported in brackets).



## 5. Internal consistency (IC) conditions

### 5.1. IC conditions based on riskless yields

If the short rate  $r_{t+1}$  is a component of  $y_t$ , for instance the first one, we have to impose an internal consistency condition implying that  $r_{t+1} = a'_1 z_t + b'_1 y_t$  is equal to the first component of  $y_t$ , that is:

$$a_1 = 0, b_1 = \tilde{e}_1,$$

where  $\tilde{e}_i$  is the vector selecting the  $i^{th}$  component of  $y_t$ .

Moreover, if another component of  $y_t$ , for instance the second one, is equal to a riskless yield of maturity  $h_0$  -ie  $R(t, h_0)$ - we have to impose that  $(1/h_0) (a'_{h_0} z_t + b'_{h_0} y_t)$  is equal to the second component of  $y_t$ , that is

$$\begin{cases} a_{h_0} &= 0 \\ b_{h_0} &= h_0 \tilde{e}_2. \end{cases}$$

### 5.2. IC conditions based on defaultable yields

Similarly, if the first component of  $x_{n,t}$  is a defaultable yield with residual maturity  $h_0$ , equation (19) implies that we have to impose:

$$\begin{cases} c_{n,h_0} &= 0 \\ f_{n,h_0} &= 0 \\ g_{n,h_0} &= h_0 \hat{e}_1. \end{cases}$$

where  $\hat{e}_i$  denotes the vector selecting the  $i^{th}$  component of  $x_{n,t}$ .

### 5.3. IC conditions based on asset returns

If the first component of  $y_t$  is the geometric return of a market index, we have to impose

$$\exp(-r_{t+1}) E_t^{\mathbb{Q}}(\exp(y_{1,t+1})) = 1.$$

Using equation (11), this gives

$$\begin{pmatrix} A_{1,0}^* & \dots & A_{J,0}^* \end{pmatrix} z_t + (\Phi_1 + \Phi_1^*) y_t = a_1' z_t + b_1' y_t,$$

with  $A_{i,0}^* = \log \left\{ \sum_{j=1}^J \pi_{ij}^* \exp [\mu_1(e_j, e_i) + \mu_1^*(e_j, e_i) + \frac{1}{2} \sigma_1^2(e_j, e_i)] \right\}$ ,  $\mu_1$  and  $\mu_1^*$  being the first components of  $\mu$  and  $\mu^*$  respectively,  $\sigma_1^2$  being the (1, 1) entry of  $\Sigma$  and  $\Phi_1$  and  $\Phi_1^*$  the first rows of  $\Phi$  and  $\Phi^*$  respectively. Then we get

$$\begin{cases} a_1 &= \begin{pmatrix} A_{1,0}^* & \dots & A_{J,0}^* \end{pmatrix}' \\ b_1 &= (\Phi_1 + \Phi_1^*)' \end{cases}$$

Similarly, if the first component of  $x_{n,t}$  is the return of a stock attached to entity  $n$ , we must have:

$$\exp(-r_{t+1}) E_t^{\mathbb{Q}}(\exp(x_{1,n,t+1})) = 1$$

or

$$r_{t+1} = \log \left[ E_t^{\mathbb{Q}}(\exp(x_{1,n,t+1})) \right].$$

Using the fact that  $(z_t, y_t, x_{n,t})$  is Car(1) under  $\mathbb{Q}$  (see Appendix A.3), it is readily seen that  $\log \left[ E_t^{\mathbb{Q}}(\exp(x_{1,n,t+1})) \right]$  is linear in  $z_t, y_t, x_{n,t}$  and the IC constraint follows.

## 6. Inference

### 6.1. Observability

We assume that  $z_t, y_t$  and the  $x_{n,t}$ 's are partitioned into  $z_t = (z_{1t}', z_{2t}')'$ ,  $y_t = (y_{1t}', y_{2t}')'$  and  $x_t = (x_{1,n,t}', x_{2,n,t}')'$ , that  $z_{1t}, y_{1t}, x_{1,n,t}$  are observed by the econometrician and  $z_{2t}, y_{2t}$  and  $x_{2,n,t}$  are not. Typically,  $z_{1,t}$  and  $z_{2,t}$  will be two regime processes valued respectively in  $E_1 = \{e_1, \dots, e_{J_1}\}$  and  $E_2 = \{e_1, \dots, e_{J_2}\}$  so  $z_t$  will be equal to  $z_{1,t} \otimes z_{2,t}$ , where  $\otimes$  denotes the Kronecker product operator. The implementation of the following estimation strategy requires that the transition probabilities do not depend on the unobserved vectors  $y_{2,t-1}$ .<sup>12</sup> Moreover, we assume that we observe at each date  $t$  a vector of risk-free yields denoted by  $R_t$  and, for each obligor  $n$ , a vector of defaultable yields denoted by  $R_{n,t}^D$ . Note that if some yields are included in the vectors  $y_t$  or  $x_{n,t}$ , they do not enter the vectors  $R_t$  and  $R_{n,t}^D$  (see Section 5). The period of observation is  $\{1, \dots, T\}$ .

### 6.2. Decomposition of the joint p.d.f. and estimation strategy

Let us denote by  $\theta^{zy}$  the vector of parameters defining the historical dynamics of  $(z_t, y_t)$ , by  $\theta_n^x$  the vector of parameters defining the conditional p.d.f. of  $x_{n,t}$  given  $\underline{z}_t, \underline{y}_t, \underline{x}_{n,t-1}$  and by  $\theta_n^d$  the vector of parameters defining the conditional p.d.f. of  $d_{n,t}$  given  $\underline{z}_t, \underline{y}_t, \underline{x}_{n,t}, \underline{d}_{n,t-1}$ .

<sup>12</sup>Formally, with the notation of Equation (1),  $p(z_t | \underline{z}_{t-1}, \underline{y}_{t-1})$  has to be equal to  $p(z_t | \underline{z}_{t-1}, \underline{y}_{1,t-1})$ .

The joint p.d.f. of  $\underline{w}_T$  is:

$$\begin{aligned} f(\underline{w}_T, \theta) &= \prod_{t=1}^T f\left(z_t, y_t \mid \underline{z}_{t-1}, \underline{y}_{t-1}; \theta^{zy}\right) \\ &\quad \times \prod_{n=1}^N \prod_{t=1}^T f\left(x_{n,t} \mid \underline{z}_t, \underline{y}_t, \underline{x}_{n,t-1}; \theta_n^x\right) \\ &\quad \times \prod_{n=1}^N \prod_{t=1}^T f\left(d_{n,t} \mid \underline{z}_t, \underline{y}_t, \underline{x}_{n,t}, \underline{d}_{n,t-1}; \theta_n^d\right). \end{aligned}$$

The parameters appearing in  $M_{t-1,t}$  are denoted by  $\theta^*$ . The theoretical values of  $R_t$  and  $R_{tn}^D$  given by the model are denoted by  $R_t(\theta^{zy}, \theta^*)$  and  $R_{nt}^D(\theta^{zy}, \theta_n^x, \theta_n^d, \theta^*)$  respectively. A sequential strategy of estimation is the following:

1. Estimate  $\theta^{zy}$  and  $\theta^*$  from the osbervations of  $y_{1t}, z_{1t}, R_t, t = 1, \dots, T$ .
2. Estimate the  $\theta_n^x$ 's and the  $\theta_n^d$ 's from the observations of  $x_{1n,t}$  and  $R_{n,t}^D, t = 1, \dots, T$ , taking as given the values of  $\theta^{zy}$  and  $\theta^*$ , and the values of  $y_{2,t}$  and  $z_{2,t}$  being fixed at the approximated values obtained from step 1.

The remaining of the current section details these two steps. The methodology that is proposed builds on the so-called inversion technique developed by Chen and Scott (1993) [15]. This technique is adapted in order to accomodate regime switching.

### 6.3. Estimation of the parameters $(\theta^{zy}, \theta^*)$

Using equation (16), we have, with obvious notations:

$$R_t(\theta^{zy}, \theta^*) = Az_t + B_1 y_{1,t} + B_2 y_{2,t}.$$

If  $m$  is the dimension of  $y_{2t}$ , let us partition  $R_t$  in  $(R'_{1,t}, R'_{2,t})'$  where  $R_{2,t}$  is of dimension  $m$ . With obvious notations, we get:

$$R_{2,t}(\theta^{zy}, \theta^*) = A_2 z_t + B_{21} y_{1,t} + B_{22} y_{2,t},$$

and denoting  $(y'_{1,t}, R'_{2,t})'$  by  $\tilde{y}_t$  we get:

$$\tilde{y}_t = \begin{pmatrix} I & 0 \\ B_{21} & B_{22} \end{pmatrix} y_t + \begin{pmatrix} 0 \\ A_2 \end{pmatrix} z_t$$

or

$$\tilde{y}_t = \tilde{B} y_t + \tilde{A} z_t$$

and

$$y_t = \tilde{B}^{-1} (\tilde{y}_t - \tilde{A} z_t)$$

and from equation (2) we get:

$$\tilde{B}^{-1} (\tilde{y}_t - \tilde{A} z_t) = \mu(z_t, z_{t-1}) + \Phi \left[ \tilde{B}^{-1} (\tilde{y}_{t-1} - \tilde{A} z_{t-1}) \right] + \Omega(z_t, z_{t-1}) \varepsilon_t$$

or

$$\tilde{y}_t = \tilde{A} z_t + \tilde{B} \mu(z_t, z_{t-1}) + \tilde{B} \Phi \left[ \tilde{B}^{-1} (\tilde{y}_{t-1} - \tilde{A} z_{t-1}) \right] + \tilde{B} \Omega(z_t, z_{t-1}) \varepsilon_t$$

or

$$\tilde{y}_t = \tilde{\mu}(z_t, z_{t-1}) + \tilde{\Phi}\tilde{y}_{t-1} + \tilde{\Omega}(z_t, z_{t-1})\varepsilon_t, \quad (24)$$

with

$$\begin{cases} \tilde{\mu}(z_t, z_{t-1}) &= \tilde{A}z_t + \tilde{B}\mu(z_t, z_{t-1}) - \tilde{B}\Phi\tilde{B}^{-1}\tilde{A}z_{t-1} \\ \tilde{\Phi} &= \tilde{B}\Phi\tilde{B}^{-1} \\ \tilde{\Omega}(z_t, z_{t-1}) &= \tilde{B}\Omega(z_t, z_{t-1}). \end{cases}$$

The conditional distribution of  $\tilde{y}_t$  given  $\underline{z}_t, \underline{\tilde{y}}_{t-1}$ , is similar to that of  $y_t$  given  $\underline{z}_t, \underline{y}_{t-1}$ , and in particular is Gaussian, the difference being that  $\tilde{y}_t$  is fully observable. Assuming moreover that the  $R_{1,t}$  are observed with Gaussian errors we get, with obvious notations:

$$\begin{aligned} R_{1,t} &= A_1 z_t + B_{11} y_{1,t} + B_{12} y_{2,t} + \xi_t \\ &= A_1 z_t + B_{11} y_{1,t} \\ &\quad + B_{12} B_{22}^{-1} (R_{2t} - A_2 z_t - B_{21} y_{1,t}) + \xi_t, \end{aligned} \quad (25)$$

with  $\xi_t \sim IIN(0, \sigma^2 I)$ .

Putting equations (24), (25) and (1) together, we have a dynamic model in which the only latent variables are  $z_{2,t}$  and which can be estimated by the maximum likelihood methods using Hamilton's approach (see Appendix B for the case when some regime variables are observed).<sup>13</sup> At this stage, IC constraints on  $(\theta^{zy}, \theta^*)$  must be taken into account.

#### 6.4. Estimation of $(\theta_n^x, \theta_n^d)$

From the inversion method of 6.3, we can get approximations of the  $y_{2,t}$ 's and smoothing algorithms provide approximations of the  $z_{2,t}$ 's (see Kim, 1994 [49]).<sup>14</sup> Then using equation (20), we get:

$$R_{t,n}^D = C_1^n z_{1,t} + C_2^n z_{2,t} + D_1^n y_{1,t} + D_2^n y_{2,t} + F_1^n x_{1,n,t} + F_2^n x_{2,n,t}. \quad (26)$$

and using equations (2), (3) and (26) and replacing  $y_{2,t}$  and  $z_{2,t}$  by their approximations, we get a system in which the only latent variables are the  $x_{2,n,t}$ . Taking  $\theta^{zy}$  and  $\theta^*$  as given, the parameters  $\theta_n^x$  and  $\theta_n^d$  can be estimated either by an inversion technique or by Kalman filtering, taking into account IC conditions.

Note that in this strategy, the observable variables  $d_{n,t}$ 's have not been used. If the recovery rate was effectively zero,  $\lambda_{n,t}$  would be the default intensity and the conditional p.d.f. of  $d_{n,t}$  given  $\underline{z}_t, \underline{y}_t, \underline{x}_{n,t}, d_{n,t-1}$  would be:

$$d_{n,t} d_{n,t-1} + (1 - d_{n,t-1}) \exp[-(1 - d_{n,t}) \lambda_{n,t}] \times [1 - \exp(-\lambda_{n,t})]^{d_{n,t}}.$$

This p.d.f. could be incorporated in the likelihood function. However, in the more realistic case of non-zero recovery rate, we have seen that (see Subsection 4.3) the  $\lambda_{n,t}$ 's must be interpreted as risk-neutral "recovery adjusted" default intensities and, therefore, they cannot be used for describing the historical dynamics of the  $d_{n,t}$ 's.

<sup>13</sup>Note that this algorithm can handle time-varying transition probabilities (which is required in the case where the  $\pi_{ij}$ 's depend on  $y_{1,t-1}$ ).

<sup>14</sup>Note that in the inversion method, the  $z_{2t}$  are replaced by those states presenting the highest smoothed probabilities.

## 6.5. Possible adaptations of the estimation strategy

Mainly for the sake of presentation clarity, the first step of the sequential strategy presented above involves only observations of macroeconomic factors and riskless yields. In particular, no credit-spread data are used in the estimation of  $\theta^*$ , the parameters appearing in the s.d.f.  $M_{t-1,t}$  as well as in the estimation of the unobserved factors  $y_{2,t}$  and of the unobserved regimes  $z_{2,t}$ . However, spread data may contain useful information for the estimation of  $\theta^{zy}$  and of  $\theta^*$ . In that case, the strategy should be adapted in order to include credit-spread data in the first step of the estimation. It can be seen that the main lines of the estimation strategy are not affected when the vector  $R_t$  and  $y_{1,t}$  considered in the first step are respectively augmented with observed defaultable-bond yields and with observable specific factor  $x_{1,n,t}$  (that are associated with the additional yields).<sup>15</sup>

Another adaptation of the strategy would be the following. The first step presented above implies a nesting of recursive computations of the theoretical formulas giving riskless (or risky) rates and recursive computation of the Kitagawa-Hamilton algorithms, which could be time-consuming. In order to alleviate the computational cost it is possible, for instance, to estimate first system (24) –or an analogue system including risky rates– with unconstrained parameters, using standard Kitagawa-Hamilton filter, and then to compute smoothed estimates values of the  $z_t$ 's. The latter values of  $z_t$  would further be considered as observations and the remaining steps would estimate all the parameters (except the ones appearing in the  $\pi_{ij,t}$ 's) using either inversion techniques or the Kalman filter.

## 7. Liquidity risk

There is compelling evidence that yields and spreads contain components that are closely linked to liquidity.<sup>16</sup> The estimation of the liquidity premium is of concern for several reasons. For instance, gauging the liquidity-risk premium provides policy makers –central bankers in particular– with insights on the valuation of liquidity by the markets (see Taylor and Williams, 2008 [60], Wu, 2008 [62] or Michaud and Upper, 2008 [56]). Furthermore, if one wants to extract default probabilities from market data, one has to distinguish between what is related to default and what is caused by the liquidity of the considered bonds.

However, the identification of the liquidity premium, that is, distinguishing between the default-related and the liquidity-related components of yield spreads, remains a challenging task. Empirical evidence points to the existence of commonality amongst the liquidity components of prices of different bonds (see e.g. Fontaine and Garcia, 2009 [34]). Therefore, the identification of the liquidity component relies on the ability to exhibit risk factors that reflects liquidity valuation. Liu, Longstaff and Mandell (2006) [52] and Feldhütter and Lando (2008) [31] develop affine term-structure models where a liquidity factor is latent and the identification is based on assumptions regarding the relative liquidity of different interest-rate instruments.<sup>17</sup> Alternatively, the liquidity factor could be proxied by some observable factors.<sup>18</sup> One may resort to intermediate –or mixed– approach, where part

<sup>15</sup>Naturally, the dimension of  $R_{2t}$  should still be equal to the number of unobserved macro-factors  $y_{2t}$ .

<sup>16</sup>The influence of liquidity effects on bond pricing has been investigated, amongst others, by Longstaff (2004) [53], Chen, Lesmond and Wei (2007) [14], Covitz and Downing (2007) [18].

<sup>17</sup>In both studies, the liquidity factor that is estimated corresponds to the so-called “convenience yield”, that can be seen as a premium that one is willing to pay when holding Treasuries. This premium stems from various features of Treasury securities, such as repo specialness (see Feldhütter and Lando, 2008).

<sup>18</sup>Amon which: bid-ask spreads, market-depth measures, bond supply, spread between bonds of the same maturity but with different ages or spread between off-the-run and on-the-run Treasuries (see, e.g., Longstaff, 2004[53] or Beber, Brandt and Kavajecz, 2009 [6]). More generally, for credit spread determ-



of the liquidity-factor dynamics is observable (through observed proxies) and part of it is latent.

Let us come back to our modeling framework. We have seen in section 4 that incorporating default risk in the pricing methodology implies to replace the short rate  $r_{t+1}$  by a “default-adjusted” short-rate  $r_{t+1} + \lambda_{n,t+1}$ . Besides, in order to take into account recovery-rate effects,  $\lambda_{n,t+1}$  can be seen as a “recovery adjusted” default intensity between  $t$  and  $t+1$  (see Appendix A.7). So the price at  $t$  of a defaultable asset providing the payoff  $g(\underline{w}_{t+h})$  at  $t+h$  in case of absence of default, is:

$$E_t^{\mathbb{Q}} [\exp(-r_{t+1} - \lambda_{n,t+1} - \dots - r_{t+h} - \lambda_{n,t+h}) g(\underline{w}_{t+h})].$$

As suggested by Duffie and Singleton (1999) [28], intensity-based model can also account for liquidity effects by introducing a stochastic process that is interpreted as the carrying cost of non-liquid defaultable securities. This process then appears alongside the default intensity in the spread between the “pure” –i.e. default and liquidity-adjusted– short rate and the short rate associated with a defaultable bond. The affine term-structure literature is relatively silent on the interpretation or the microfoundations of the illiquidity intensity. In a theoretical paper analyzing interactions between credit and liquidity risks, He and Xiong (2011) [44] show that such an illiquidity intensity may reflect the probability of occurrence of a liquidity shock; upon the arrival of this shock, the bond investor has to exit by selling his bond at a fractional cost (i.e. the selling price is equal to a fraction of the price that would have prevailed in the absence of the liquidity shock); the fractional cost is the analogous to the fractional loss  $(1 - \zeta)$  in the default case (see also Ericsson and Renault, 2006 [30] for a similar interpretation). Accordingly, let us introduce an “illiquidity intensity” between  $t$  and  $t+1$ , denoted with  $\lambda_{n,t+1}^L$ . If  $\lambda_{n,t+1}$  and  $\lambda_{n,t+1}^L$  are specified in an affine way,

$$\begin{cases} \lambda_{n,t+1} &= \alpha'_n z_{t+1} + \beta'_n y_{t+1} + \gamma'_n x_{n,t+1} \\ \lambda_{n,t+1}^L &= \alpha_n^{L'} z_{t+1} + \beta_n^{L'} y_{t+1} + \gamma_n^{L'} x_{n,t+1}, \end{cases}$$

we could price not only riskless bonds  $B_n(t, h)$  and defaultable bonds  $B_n^D(t, h)$  as above, but also bonds facing liquidity risk  $B_n^L(t, h)$  and bonds facing both default and liquidity risk  $B_n^{DL}(t, h)$ . We would have:

$$\begin{cases} B(t, h) &= E_t^{\mathbb{Q}} [\exp(-r_{t+1} - \dots - r_{t+h})] \\ B_n^D(t, h) &= E_t^{\mathbb{Q}} [\exp(-r_{t+1} - \lambda_{n,t+1} - \dots - r_{t+h} - \lambda_{n,t+h})] \\ B_n^L(t, h) &= E_t^{\mathbb{Q}} \left[ \exp \left( -r_{t+1} - \lambda_{n,t+1}^L - \dots - r_{t+h} - \lambda_{n,t+h}^L \right) \right] \\ B_n^{DL}(t, h) &= E_t^{\mathbb{Q}} \left[ \exp \left( -r_{t+1} - \lambda_{n,t+1} - \lambda_{n,t+1}^L - \dots - r_{t+h} - \lambda_{n,t+h} - \lambda_{n,t+h}^L \right) \right]. \end{cases}$$

In the context of a Car(1) risk-neutral dynamics of  $(z_t, y_t, x_{n,t})$ , these prices are exponential linear in  $(z_t, y_t, x_{n,t})$  and the corresponding yields are linear in  $(z_t, y_t, x_{n,t})$ .

If the obligors issue only bonds facing both default and liquidity risks, and if the same factors affect both kinds of intensities, it is not possible to distinguish between the two of them. In order to operate –or to gain some insights on– a decomposition between the default intensity on the one hand and the liquidity intensity on the other, one has to rely on additional assumptions. For instance, these assumptions may reflect some priors about the relative effects of the risk factors on the different obligors or on the different types of securities (as in Liu, Longstaff and Mandell (2006) [52] and Feldhütter and Lando (2008) [31]).

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inants, see e.g., Elton (2001) [29] or Collin-Dufresne, Goldstein and Martin (2001) [17].

## 8. Model extensions

### 8.1. Multi-lag dynamics for $y_t$ and $x_{n,t}$ processes

The model can easily be extended to allow for  $y_t$  and  $x_{n,t}$  dynamics that include several lags. In particular, when observed data are used in the estimation process –the  $y_{1,t}$  and  $x_{1,n,t}$  defined in Section 6–, preliminary analysis of the data could point to the need of taking different lags into account to model the historical dynamics of these variables. The flexibility in the choice of the lag structure constitutes an advantage of working in discrete-time over most continuous-time models (see, e.g., Monfort and Pegoraro, 2007 [57] or Gouriéroux, Monfort and Polimenis, 2006 [38]).

Equations (2) and (3) imply that the multivariate factors  $y_t$  and  $x_t$  follow auto-regressive process of order one. However, to the extent that a VAR( $p$ ) amounts to a VAR(1) once the last  $p$  lags of the endogenous variable are stacked in the same vector, the pricing techniques of the bonds –namely equations (16) and (20)– are not affected if  $y_t$  and  $x_t$  follow VAR( $p$ ). However, in order to make the estimation strategy presented in Section 6 still effective –in particular regarding inversion techniques–, the unobserved vector variables  $y_{2,t}$  and  $x_{2,n,t}$  should not enter equations (2) and (3) with lags larger than one. To the extent that this restriction only applies to the unobserved factors –for which insights on the appropriate distributions are a priori not readily available– such a constraint is not really restrictive.

### 8.2. Interpretation of a regime as the default state of an entity

In this subsection, we consider the specific case where one the Markov chain included in  $z_t$  corresponds to the default state of a given entity.<sup>19</sup> The specificity of that situation lies in the fact that the default of this entity then enters the s.d.f.. Therefore, we leave the framework described in Subsection 3.1 where all defaultable entities were small enough to have no impact at the macroeconomic level. As a consequence, the “zero” entity may represent a whole industry or a very big institution. This could be extended to a few major entities but one has to bear in mind that increasing their number results in an exponential growth in the dimension of  $z_t$ .

The fact that this default enter the s.d.f. results in a new component in bond prices: a compensation for investors risk-aversion towards the default event of entity zero. As pointed out by Yu (2002) [63] and Jarrow, Lando and Yu (2005) [46], such components arise only when the default-event risk is not diversifiable.

As mentioned in Section 1, this interpretation is also linked with previous studies attempting to introduce contagion effects in affine term-structure models. Indeed, the default of entity zero may lead to a simultaneous increase in the default intensities of any other debtor (through the regime variable  $z_t$  that may enter all default intensities).

For sake of simplicity, let us assume that such a crisis variable is the only regime captured by  $z_t$ , which can be observable or not. In this case, assuming that the state  $e_2 = (0, 1)'$  is the absorbing crisis state, we have:

$$\begin{aligned}\pi(e_2 | e_2, y_{t-1}) &= 1 \\ \pi(e_1 | e_2, y_{t-1}) &= 0.\end{aligned}$$

Moreover, we could specify:

$$\pi(e_1 | e_1, y_{t-1}) = \exp(-\lambda_{0,t-1}),$$

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<sup>19</sup>We can deal with several Markov chains by writing vector  $z_t$  as a Kronecker product of several chains.

with  $\lambda_{0,t-1} = \alpha_0 + \beta'_0 y_{1,t-1}$ . In this case,  $\lambda_{0,t-1}$  can be interpreted as a systemic-risk intensity. Conditions (10)  $\{\pi(e_j | e_i, y_{t-1}) \exp[\delta_j(e_i, y_{t-1})] = \pi_{ij}^*\}$  imply the followings:

- $\pi_{21}^* = 0$ ,  $\pi_{22}^* = 1$ ,  $\delta_1(e_2, y_{t-1})$  is undefined,  $\delta_2(e_2, y_{t-1}) = 0$  and, therefore,  $\delta'(e_2, y_{t-1}) z_t = 0$ .
- $\exp[\delta_1(e_1, y_{t-1})] = \pi_{11}^* \exp(\lambda_{0,t-1})$  or  $\delta_1(e_1, y_{t-1}) = \log \pi_{11}^* + \alpha_0 + \beta'_0 y_{1,t-1}$ .
- $\exp[\delta_2(e_1, y_{t-1})] = (1 - \pi_{11}^*) [1 - \exp(-\lambda_{0,t-1})]^{-1}$ , or  $\delta_2(e_1, y_{t-1}) = \log(1 - \pi_{11}^*) - \log[1 - \exp(-\alpha_0 - \beta'_0 y_{1,t-1})]$ .

Denoting  $\pi_{11}^* = \exp(-\lambda_0^*)$ ,  $\lambda_0^*$  being the systemic-risk intensity in the risk-neutral world, we get:

$$\begin{aligned} \delta_1(e_1, y_{t-1}) &= \lambda_{0,t-1} - \lambda_0^* \\ \delta_2(e_1, y_{t-1}) &= \log[1 - \exp(-\lambda_0^*)] - \log[1 - \exp(-\lambda_{0,t-1})] \\ &\simeq \log(\lambda_0^*) - \log(\lambda_{0,t-1}) \text{ if } \lambda_0^*, \lambda_{0,t-1} \text{ are small.} \end{aligned}$$

In particular, the risk-neutral intensity  $\lambda_0^*$  and the historical intensity  $\lambda_{0,t-1}$  are different functions, contrary to what happened in the previous sections. Both the riskless yields:

$$R(t, h) = \frac{1}{h} (a'_h z_t + b'_h y_t)$$

and the defaultable yields:

$$R_n^D(t, h) = \frac{1}{h} (c'_{n,h} z_t + f'_{n,h} y_t + g'_{n,h} x_{n,t})$$

will be different functions of  $y_t$  (and of  $x_{nt}$  for  $R_n^D(t, h)$ ) before and after the systemic crisis. The term structure of the impact of the systemic crisis will be:

$$\begin{cases} a_{2,h} - a_{1,h} & \text{for the riskless yield of residual maturity } h, \\ c_{2,n,h} - c_{1,n,h} & \text{for the defaultable yield of residual maturity } h, \text{ for the } n^{th} \text{ entity.} \end{cases}$$

### 8.3. A sector-contagion model

#### 8.3.1. General approach

In this subsection, we propose another specific use of the regimes that makes it possible to model sector-contagion phenomena. As explained in the introduction, our assumptions prevent us from making the default intensity of any entity depend on the default event of other entities. In other words, the baseline framework does not allow us to account for contagion at the debtor level (except in the specific case presented in 8.2). Nevertheless, as shown here, this can be done at a sector level, the sectors representing for instance different industries or different geographical areas.

Specifically, in this model, each debtor belongs to one of the sectors. At each period, a sector is either “infected” or not infected. When a sector is infected, the default intensities of its constituent entities tend to be higher. Let us denote by  $S_{i,t}$  the state the  $i^{th}$  sector at time  $t$ :  $S_{i,t}$  is equal to  $[1, 0]'$  if the  $i^{th}$  sector is infected at time  $t$ , and is equal to  $[0, 1]'$  otherwise. If we have  $N_S$  sectors, then we have to consider  $2^{N_S}$  regimes, the regime variable  $z_t$  being given by:

$$z_t = S_{1,t} \otimes S_{2,t} \otimes \dots \otimes S_{N_S,t}$$

where  $\otimes$  denotes the Kronecker product. In such a model, one can make the default intensity of any firm depend on the state of the sectors (and, in particular, on the state of its own sector). Further, the sector-contagion phenomena can be obtained through the specifications of the regime-transition matrix. Indeed, this matrix contains the probabilities that any sector gets infected (or cured) given the states of the other sectors.

### 8.3.2. Numerical example

In this example, the risk-free short-term rate  $r_{t+1}$  has the same historical and risk-neutral dynamics of as in 4.4. We consider three homogeneous sectors. The probability that a sector gets cured/infected at time  $t$  depends on the number of infected sectors at the previous period. In that case, the regime-transition matrix is defined by a set of probabilities like the one reported in Table 2. In our example, the probability of getting infected is far higher when at least one sector is already infected than when none of them is infected.

Table 2 – **Probabilities of getting infected/cured**

| Number of infected sectors ( $\sum_i [0, 1] \times S_{i,t}$ ) | 0     | 1   | 2   | 3   |
|---|-------|-----|-----|-----|
| Probability of getting infected (in $t + 1$ )                 | 0.25% | 10% | 10% | NA  |
| Probability of getting cured (in $t + 1$ )                    | NA    | 10% | 10% | 10% |

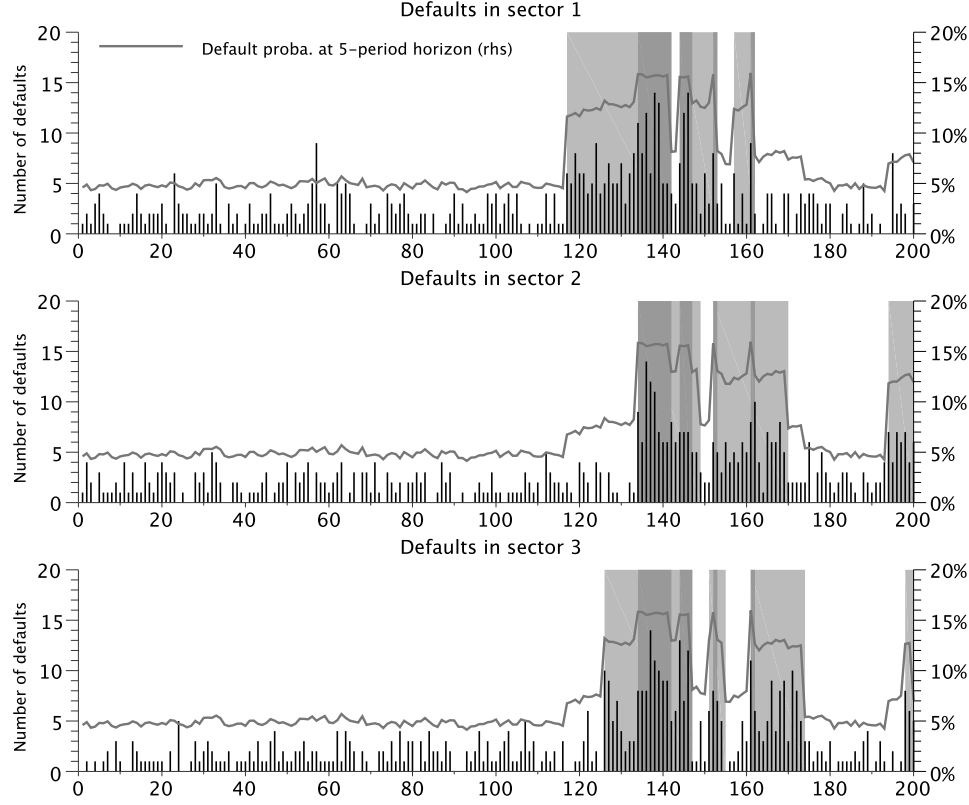
The default intensities of sector- $i$  firms are given by:

$$\lambda_{i,t} = 0.01 + 0.02 \times \mathbb{I}_{\{S_t^i=1\}} + 0.02 \times \mathbb{I}_{\{S_t^1=1\}} \mathbb{I}_{\{S_t^2=1\}} \mathbb{I}_{\{S_t^3=1\}} + 0.002 y_{r,t},$$

which implies that the default intensity of a Sector- $i$  entity increases by two percentage points when Sector  $i$  gets infected and increases by an additional two percentage points if all sectors become infected simultaneously.

Figure 5: **Simulated sample of the sector-contagion model**

**Notes:** Each panel corresponds to one sector. There are 600 debtors in the portfolio (200 per sector). The vertical bars represent the number of firms that have defaulted during the considered period. At the end of each period, defaulted firms are replaced by new ones (of the same sector). Grey-shaded areas indicate periods during which the considered sector is in distress. Darker areas indicate periods when all three sectors are in distress.



Let us now consider a portfolio of 600 debtors, with 200 debtors in each sector. Figure 5 shows a simulation of the timing of defaults for this portfolio. Each panel corresponds to one of the three sectors. At one point, Sector 1 gets infected (see the grey area in the first panel of Figure 5). While the default intensities of Sector-2 and Sector-3 firms are not contemporaneously impacted by the infection of the first sector, 5-year default probabilities of Sector-2 and Sector-3 firms shift upwards. This is accounted for by the fact that once Sector 1 is infected, the probability that Sector 2 and Sector 3 get infected over the next periods is higher. A few periods later, Sector 3 and then Sector 2 get infected.

#### 8.4. Modeling credit-rating transitions

In their seminal study of credit spread, Jarrow, Lando and Turnbull (1997) [45] model rating transitions as a time-homogenous Markov chain. That is, in their model, whether a firm's rating will change in the next period depends on its current rating only and the probability of changing from one rating to the other remains the same over time. Different studies suggest however that –per-period– transition probabilities are time-varying (see e.g. Lucas and Lonski, 1992 [54] or Feng, Gourioux and Jasiak, 2008 [32]).

In the present subsection, we show how our framework can be adapted in order to account explicitly for rating migration. Building on Lando's (1998) [50] approach (see also Feldhütter and Lando, 2008 [31]), the structure accommodates a time-varying rating-migration

matrix while allowing different ratings to respond in a correlated yet different fashion to the same change in the general economic conditions. The time variability of the rating-migration probabilities results from Gaussian shocks as well as from regime shifts.

### 8.4.1. Adaptation of the framework

While most of the previous framework is still valid, some changes regard the modeling of the default intensity. Specifically, the historical dynamics of  $(z_t, y_t, x_{n,t})$ , as well as the s.d.f. specifications are still given by equations (1), (2), (3) and (6). However, in this adapted framework, each firm  $n$  is also characterized by a credit-rating process, denoted by  $\tau_{n,t}$ . For any firm  $n$  and period  $t$ ,  $\tau_{n,t}$  can take one of  $K$  values: the first  $K - 1$  values correspond to credit ratings and the  $K^{th}$  corresponds to the default state.<sup>20</sup> Like the  $d_{n,t}$ 's, the  $\tau_{n,t}$ 's,  $n = 1, \dots, N$ , are independent conditionally to  $(z_t, y_t, x_t, \underline{y}_{t-1})$ . In addition, we assume that the rating transition probabilities, for firm  $n$  and from period  $t - 1$  to period  $t$ , is a function of  $(z_t, y_t, x_{n,t})$ . Accordingly, this transition matrix is denoted with  $\Pi(z_t, y_t, x_{n,t})$  and we have:

$$P(\tau_{n,t} = j \mid \tau_{n,t-1} = i) = \Pi_{i,j}(z_t, y_t, x_{n,t}),$$

where  $\Pi_{i,j}(z_t, y_t, x_{n,t})$ , the  $(i, j)$  entry of  $\Pi(z_t, y_t, x_{n,t})$ , represents the actual probability of going from state  $i$  to state  $j$  in one time step. Each of these entries must be in  $[0, 1]$  and for each line, the sum of the entries must sum to one. In other words,  $\begin{bmatrix} 1 & \dots & 1 \end{bmatrix}'$  is an eigenvector of  $\Pi(z_t, y_t, x_{n,t})$  associated with the eigenvalue 1. In addition, the default state being absorbing, the bottom row of  $\Pi(z_t, y_t, x_{n,t})$  is equal to  $\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}$ .

In this context, a defaultable zero-coupon bond providing one money unit at  $t + h$  if entity  $n$  is still alive in  $t + h$  and zero otherwise has a price, in period  $t$ , that is given by (assuming that entity  $n$  has not defaulted before  $t$ ):

$$B_n^D(t, h) = E_t^{\mathbb{Q}} \left[ \exp(-r_{t+1} - \dots - r_{t+h}) \mathbb{I}_{\{\tau_{n,t+h} < K\}} \right]. \quad (27)$$

In order to keep a quasi-explicit formula for defaultable zero-coupon bonds, we assume that  $\Pi(z_t, y_t, x_{n,t})$  admits the diagonal representation:

$$\Pi(z_t, y_t, x_{n,t}) = V \cdot \Psi(z_t, y_t, x_{n,t}) \cdot V^{-1},$$

where the columns of  $V$  are the eigenvectors of  $\Pi(z_t, y_t, x_{n,t})$  and constitute a basis in  $\mathbb{R}^K$  and  $\Psi(z_t, y_t, x_{n,t})$  is a diagonal matrix of eigenvalues that are positive and smaller than one.<sup>21</sup> Given that 1 is an eigenvalue of  $\Pi(z_t, y_t, x_{n,t})$ ,  $\Psi(z_t, y_t, x_{n,t})$  can be written in the following manner:

$$\Psi(z_t, y_t, x_{n,t}) = \begin{bmatrix} \exp[-\psi_1(w_t)] & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \exp[-\psi_{K-1}(w_t)] & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix},$$

with, for any  $i < K$ ,  $\psi_i(w_t) \geq 0$ . Then, it is easily seen that, conditionally on  $(\underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, \tau_{n,t} = i)$  the probability of defaulting before  $t + h$  corresponds to the entry  $(i, K)$  of

<sup>20</sup>For instance, rating 1 can be the highest (Aaa in Moody's rankings) and  $K-1$  can be the lowest (C in Moody's rankings). In addition, we have,  $d_{n,t} = \mathbb{I}(\tau_{n,t} = K)$ .

<sup>21</sup>The fact that the eigenvalues have a modulus smaller than one is necessary in the case of time-homogenous Markov chain processes.

the matrix that is given by:

$$V \cdot \Psi(z_{t+1}, y_{t+1}, x_{n,t+1}) \dots \Psi(z_{t+h}, y_{t+h}, x_{n,t+h}) \cdot V^{-1}.$$

This probability is therefore given by:

$$P(\tau_{n,t+h} = K \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, \tau_{n,t} = i) = \sum_{j=1}^K V_{i,j} V_{j,K}^{-1} \exp \left[ - \sum_{p=1}^h \psi_j(w_{t+p}) \right],$$

where  $V_{i,j}$  and  $V_{i,j}^{-1}$  are the entries  $(i, j)$  of  $V$  and  $V^{-1}$ . Since  $V_{i,K} V_{K,K}^{-1} = 1$  (see Appendix C) using  $\psi_K \equiv 0$ , we get:

$$P(\tau_{n,t+h} < K \mid \underline{z}_{t+h}, \underline{y}_{t+h}, \underline{x}_{n,t+h}, \tau_{n,t} = i) = - \sum_{j=1}^{K-1} V_{i,j} V_{j,K}^{-1} \exp \left[ - \sum_{p=1}^h \psi_j(w_{t+p}) \right]. \quad (28)$$

If the eigenvalues  $\psi_j$  are some linear combinations of  $(z_t, y_t, x_{n,t})$ , Equations (27) and (28) imply that the price of a bond is a sum of  $K - 1$  multi-horizon Laplace transforms. As a consequence, the bond prices can be obtained using the algorithm presented in Lemma 2. However, it should be noted that in this context, the prices are no longer exponential affine in the factors, which implies in particular that the Kalman filter has to be adapted so as to accomodate the nonlinearity of the state-space measurement equations. In such a context, Feldhütter and Lando (2008) [31] use the extended Kalman filter. As an alternative, the unscented Kalman filter can be implemented.

#### 8.4.2. Numerical example

Let us consider again the processes  $r_t$  and  $z_t$  as specified in 4.4. In the present model, the credit-migration matrices are of the form:

$$\Pi(z_t, y_t, x_{n,t}) = V \cdot \begin{bmatrix} \exp[-\alpha_1 z_t - \beta_1 y_{r,t}] & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \exp[-\alpha_{K-1} z_t - \beta_{K-1} y_{r,t}] & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \cdot V^{-1}$$

In order to get plausible matrices, the first-regime calibration –that involves the  $\alpha_{i,1}$ ’s– is based on the one-year-average rating-migration matrix for European corporates provided by Moody’s (Moody’s, 2010 [59]). This matrix is reported in Table 3. The spectral decomposition of this matrix provides us with the matrix of eigenvectors  $V$ . The eigenvalues are real and comprised between 0 and 1. Accordingly, they are of the form  $\exp(-\alpha_{i,1})$ . The  $\alpha_{i,1}$  are reported in Table 4. The definition of the second regime requires a second set of  $\alpha_i$ ’s, denoted by  $\{\alpha_{i,2}\}_{i=1 \dots K-1}$ . We calibrate the latter in order to have 5-year default probabilities that are higher than those obtained with the first-regime transition matrix (see Table 4).<sup>22</sup> Finally, the  $\beta_i$ ’s are given by  $(\alpha_{i,1} - \alpha_{i,2})/5$ .

Table 3 – **Baseline matrix of rating-migration probabilities**

Notes: This matrix is based on Moody’s (2010) [59] (Exhibit 12: One-year average ratings-transition for European corporates 1985-2009). According to the industry standard, the probability of transitions to the “not rated” state is distributed among all states in proportion to their values (see Bangia et al., 2002 [4]).

<sup>22</sup>The 5-year default probabilities are computed conditionally on the absence of regime switching (i.e. as if the current regime is to last 5 years).

|       | Aaa   | Aa    | A     | Baa   | Ba    | B     | Caa-C | Default |
|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| Aaa   | 0.911 | 0.084 | 0.004 | 0.000 | 0.001 | 0.000 | 0.000 | 0.000   |
| Aa    | 0.009 | 0.902 | 0.083 | 0.005 | 0.000 | 0.000 | 0.000 | 0.000   |
| A     | 0.000 | 0.042 | 0.898 | 0.055 | 0.003 | 0.000 | 0.000 | 0.001   |
| Baa   | 0.000 | 0.004 | 0.072 | 0.868 | 0.041 | 0.009 | 0.003 | 0.001   |
| Ba    | 0.000 | 0.000 | 0.007 | 0.074 | 0.788 | 0.107 | 0.012 | 0.011   |
| B     | 0.000 | 0.000 | 0.004 | 0.004 | 0.073 | 0.794 | 0.092 | 0.033   |
| Caa-C | 0.000 | 0.003 | 0.001 | 0.000 | 0.007 | 0.106 | 0.706 | 0.177   |

Table 4 – **Eigenvalues of the transition matrix under both regimes**

Notes: “Regime 1” is consistent with the transition matrix reported in Table 3. Regime 2 is intended to depict a “crisis” regime. The  $\alpha_{i,j}$ ’s ( $i = 1, \dots, 7, j = 1, 2$ ) are such that the  $\exp(-\alpha_{i,j})$ ’s are the eigenvalues –those different from 1– of the rating-transition matrix obtained under regime  $j$  (when  $y_{r,t} = 0$ ).

| 5-yr default prob.                       | Aaa             | Aa              | A               | Baa             | Ba              | B               | Caa-C           |
|--|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Regime 1                                 | 0.057%          | 0.24%           | 0.80%           | 1.91 %          | 8.72%           | 21.8%           | 52.0%           |
| Regime 2                                 | 0.774 %         | 1.79 %          | 3.01%           | 6.40%           | 16.74%          | 32.6%           | 63.2%           |
| $-\log(\text{eigenvalues})$              | 1 <sup>st</sup> | 2 <sup>nd</sup> | 3 <sup>rd</sup> | 4 <sup>th</sup> | 5 <sup>th</sup> | 6 <sup>th</sup> | 7 <sup>th</sup> |
| $\alpha_{i,1}$ ( $i = 1, \dots, K - 1$ ) | 0.009           | 0.069           | 0.097           | 0.143           | 0.213           | 0.311           | 0.464           |
| $\alpha_{i,2}$ ( $i = 1, \dots, K - 1$ ) | 0.017           | 0.110           | 0.146           | 0.205           | 0.294           | 0.463           | 0.807           |

Figure 6 displays yield curves for selected ratings under both regimes (for  $y_{r,t} = 0$ ). Figure 7 presents some simulation results. The upper panel shows the time fluctuations of downgrade probabilities for two different ratings. The lower panel displays yield spreads between 10-year zero-coupon bonds issued by A-rated or Baa-rated firms and 10-year zero-coupon bonds issued by Aaa-rated firms.

Figure 6: **Yield curves for selected ratings (with impact of regimes)**

Notes: The left plot shows yield curves for selected ratings, with  $y_{r,t} = 0$  and  $z_t = [1, 0]'$  (solid lines) or  $z_t = [0, 1]'$  (dashed lines). The right plot shows the term structure of spreads vs. Aaa-rated bonds.

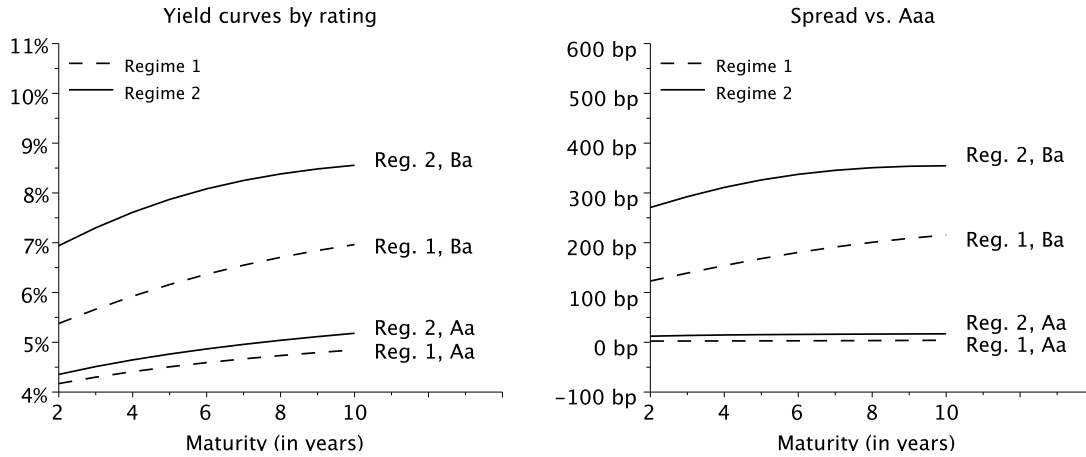
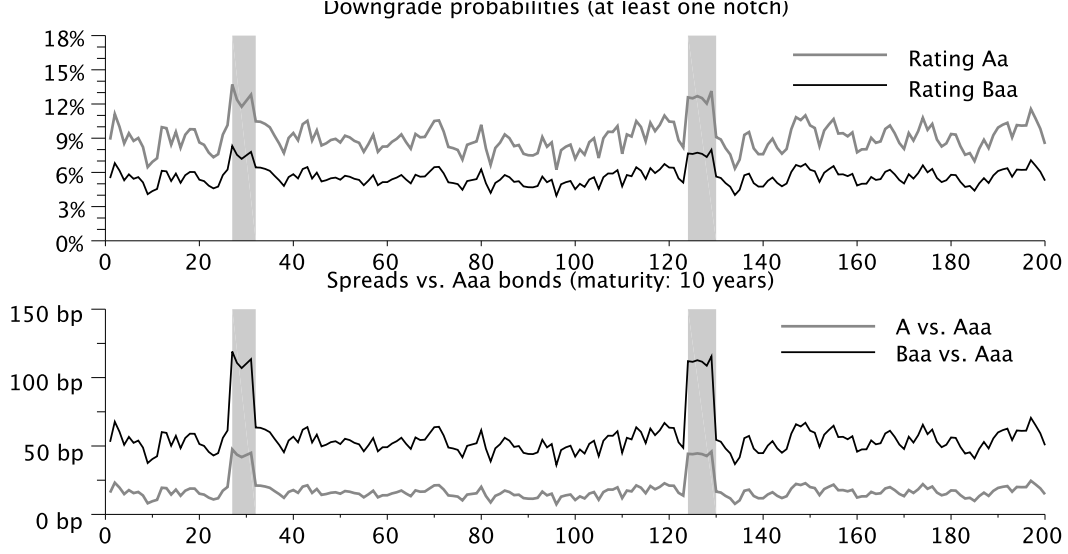




Figure 7: **Simulated downgrade probabilities and spreads**

Notes: The lower plot shows simulated downgrade probabilities for two ratings (the downgrade can be of one or more notches). Formally, for rating  $j$ , the upper panel plots  $P(\tau_{n,t} > \tau_{n,t-1} \mid \mathbf{z}_t, \mathbf{y}_t, \mathbf{x}_{n,t}, \tau_{n,t-1} = j)$ . The grey-shaded areas indicate “crisis” periods. The lower plot shows the yield spreads between 10-year zero-coupon bonds issued by A-rated or Baa-rated debtors and zero-coupon bonds issued by Aaa-rated issuers.



## 9. Conclusion

In this paper, we have proposed an econometric framework aimed at jointly modeling yield curves associated with different defaultable issuers. Default intensities and yields are affine functions of a multivariate process which is Compound autoregressive (Car) in the risk-neutral world and thus provides us with quasi-explicit (recursive) formulas for both risk-free and defaultable bond prices.

The risk factors follow discrete-time conditionally Gaussian processes, with drifts and variance-covariance matrices that are subject to regime shifts described by a Markov chain with (historical) non-homogenous transition probabilities. The regime-switching feature is relevant for credit models in several respects. First, it makes it possible to capture non-linear behaviors of yields and spreads, which is consistent with empirical evidence. Second, it is appropriate to capture default clusters. Third, it offers some ways of dealing with specific forms of contagion. To that respect, we show how the framework can be used to capture sector-contagion phenomena. An other extension accommodates credit-rating migrations. While flexible, the model remains tractable and amenable to empirical estimation. To that end, a sequential estimation strategy is proposed in the paper.

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## A. Proofs of Sections 3 and 4

### A.1. Proof of Proposition 2

$$\begin{aligned}
\varphi_{t-1}^{\mathbb{Q}}(u, v) &= E_{t-1}^{\mathbb{Q}}(\exp[u'z_t + v'y_t]) \\
&= E_{t-1}\left(\exp\left[-\frac{1}{2}\nu_t'\nu_t + \nu_t'\varepsilon_t + \delta_{t-1}'z_t + u'z_t + v'y_t\right]\right) \\
&= \exp(v'\Phi y_{t-1}) \times \\
&\quad E_{t-1}\left(\exp\left[-\frac{1}{2}\nu_t'\nu_t + \nu_t'\varepsilon_t + \delta_{t-1}'z_t + u'z_t + v'\mu_t + v'\Omega_t\varepsilon_t\right]\right) \\
&= \exp(v'\Phi y_{t-1}) \times \\
&\quad E_{t-1}\left(\exp\left[-\frac{1}{2}\nu_t'\nu_t + \frac{1}{2}(\nu_t' + v'\Omega_t)(\nu_t' + v'\Omega_t)' + v'\mu_t + u'z_t + \delta_{t-1}'z_t\right]\right) \\
&= \exp(v'\Phi y_{t-1}) E_{t-1}\left(\exp\left[v'\Omega_t\nu_t + \frac{1}{2}v'\Sigma_t v + v'\mu_t + u'z_t + \delta_{t-1}'z_t\right]\right).
\end{aligned}$$

Using the expression given for  $A_{i,t-1}(u, v)$  leads to the result.

### A.2. P.d.f. under the risk-neutral world (Proof of Lemma 1)

Let us consider a couple  $(X, Y)$  of multivariate random vectors. Let denote with  $f^{\mathbb{H}}(X, Y)$  and  $f^{\mathbb{Q}}(X, Y)$  their respective joint p.d.f. under the probability measure  $\mathbb{H}$  and  $\mathbb{Q}$  and assume that the Radon-Nikodym derivative that relates  $\mathbb{H}$  and  $\mathbb{Q}$  depends on  $X$  only and is proportional to  $M(X)$ . We have:

$$\begin{aligned}
f^{\mathbb{Q}}(X, Y) &= \frac{f^{\mathbb{H}}(X, Y)M(X)}{\int f^{\mathbb{H}}(X, Y)M(X)dXdY} \\
&= \frac{f^{\mathbb{H}}(X)f^{\mathbb{H}}(Y | X)M(X)}{\iint f^{\mathbb{H}}(X)f^{\mathbb{H}}(Y | X)M(X)dXdY} \\
&= \frac{f^{\mathbb{H}}(X)f^{\mathbb{H}}(Y | X)M(X)}{\int f^{\mathbb{H}}(X)M(X) \left[\int f^{\mathbb{H}}(Y | X)dY\right] dX} \\
&= \frac{f^{\mathbb{H}}(X)M(X)}{\int f^{\mathbb{H}}(X)M(X)dX} f^{\mathbb{H}}(Y | X) \\
&= f^{\mathbb{Q}}(X)f^{\mathbb{H}}(Y | X).
\end{aligned}$$

### A.3. The risk-neutral Laplace transform of $(z_t, y_t, x_{n,t})$

In this appendix, we compute  $E_{t-1}^{\mathbb{Q}}(\exp[u'z_t + v'y_t + w'x_{n,t}])$  and show that it is exponential affine in  $(z_{t-1}, y_{t-1}, x_{n,t-1})$ , that is, we show that  $(z_t, y_t, x_{n,t})$  is Car(1) (see Darolles, Gouriou and Jasiak, 2006 [20]).

$$\begin{aligned}
E_{t-1}^{\mathbb{Q}}(\exp[u'z_t + v'y_t + w'x_{n,t}]) &= E_{t-1}^{\mathbb{Q}}(\exp[u'z_t + v'y_t + w'(q_{1n}(z_t, z_{t-1}) + \\
&\quad Q_{2n}y_t + Q_{3n}y_{t-1} + Q_{4n}x_{n,t-1} + Q_{5n}(z_t, z_{t-1})\eta_{n,t}])) \\
&= \exp(w'Q_{3n}y_{t-1} + w'Q_{4n}x_{n,t-1}) \times \\
&\quad E_{t-1}^{\mathbb{Q}}(\exp[u'z_t + (v' + w'Q_{2n})y_t + \\
&\quad w'q_{1n}(z_t, z_{t-1}) + w'Q_{5n}(z_t, z_{t-1})\eta_{n,t}])) \\
&= \exp(w'Q_{3n}y_{t-1} + w'Q_{4n}x_{n,t-1}) \times \\
&\quad E_{t-1}^{\mathbb{Q}}(\exp[u'z_t + w'q_{1n}(z_t, z_{t-1}) + w'Q_{5n}(z_t, z_{t-1})\eta_{n,t} + \\
&\quad (v' + w'Q_{2n})(\mu_t + \mu_t^*) + (\Phi + \Phi^*)y_{t-1} + \Omega_t\varepsilon_t^*])) \\
&= \exp[\{(v' + w'Q_{2n})(\Phi + \Phi^*) + w'Q_{3n}\}y_{t-1} + \\
&\quad w'Q_{4n}x_{n,t-1} + (\tilde{A}_1(u, v, w) \dots \tilde{A}_J(u, v, w))z_{t-1}]
\end{aligned}$$

with

$$\begin{aligned}
\tilde{A}_i(u, v, w) &= \log\left(\sum_{j=1}^J \pi_{ij}^* \exp\{u_j + (v' + w'Q_{2n})[\mu(e_j, e_i) + \mu^*(e_j, e_i)] + w'q_{1n}(e_j, e_i) + \right. \\
&\quad \left. \frac{1}{2}(v' + w'Q_{2n})\Sigma(e_j, e_i)(v + Q'_{2n}w) + \frac{1}{2}w'Q_{5n}(e_j, e_i)Q'_{5n}(e_j, e_i)w\}\right).
\end{aligned}$$

The fact that  $(z_t, y_t, x_{n,t}, d_{n,t})$  is not Car(1) is obtained by noting that (for  $d_{n,t-1} = 0$ ):

$$\begin{aligned}
E_{t-1}^{\mathbb{Q}}(\exp[u'z_t + v'y_t + w'x_{n,t} + sd_{n,t}]) &= \\
E_{t-1}^{\mathbb{Q}}(E(\exp[u'z_t + v'y_t + w'x_{n,t} + sd_{n,t}] \mid z_t, y_t, x_{n,t}, d_{n,t-1} = 0)) &= \\
E_{t-1}^{\mathbb{Q}}(\exp[u'z_t + v'y_t + w'x_{n,t}] E(\exp[sd_{n,t}] \mid z_t, y_t, x_{n,t}, d_{n,t-1} = 0)) &= \\
E_{t-1}^{\mathbb{Q}}(\exp[u'z_t + v'y_t + w'x_{n,t}] (\exp(-\lambda_{n,t}) + [1 - \exp(-\lambda_{n,t})]\exp(s))) &=
\end{aligned}$$

This shows that  $E_{t-1}^{\mathbb{Q}}(\exp[u'z_t + v'y_t + w'x_{n,t} + sd_{n,t}])$  will be only a *sum* of two terms that are exponential affine in  $(z_{t-1}, y_{t-1}, x_{n,t-1}, d_{n,t-1})$ . Consequently,  $(z_t, y_t, x_{n,t}, d_{n,t})$  is not Car(1).

#### A.4. Proof of Lemma 2

The formula is true for  $h = 1$  since:

$$L_{t,1}(\omega) = E_t(\omega'_H Z_{t+1}) = \exp[a'(\omega_H)Z_t + b(\omega_H)]$$

and therefore  $A_1 = a(\omega_H)$  and  $B_1 = b(\omega_H)$ .

if it is true for  $h - 1$ , we get:

$$\begin{aligned}
L_{t,h}(\omega) &= E_t[\exp(\omega'_{H-h+1}Z_{t+1}) E_{t+1}(\omega'_{H-h+2}Z_{t+2} + \dots + \omega'_H Z_{t+H})] \\
&= E_t[\exp(\omega'_{H-h+1}Z_{t+1}) L_{t+1,h-1}(\omega)] \\
&= E_t[\exp(\omega'_{H-h+1}Z_{t+1} + A_{h-1}Z_{t+1} + B_{h-1})] \\
&= \exp[a(\omega'_{H-h+1} + A_{h-1})Z_t + b(\omega'_{H-h+1} + A_{h-1}) + B_{h-1}]
\end{aligned}$$

and the result follows.

#### A.5. Proof of Proposition 3

We have:

$$B(t, h) = \exp(-a'_1 z_t - b'_1 y_t) E_t^{\mathbb{Q}}(-a'_1 z_{t+1} - b'_1 y_{t+1} - \dots - a'_1 z_{t+h-1} - b'_1 y_{t+h-1}).$$

Using Lemma 2 with  $\omega_H = 0$ ,  $\omega'_h = (-a'_1, -b'_1)$  for  $h = 1, \dots, H-1$ , we get:

$$B(t, h) = \exp \left( -a'_1 z_t - b'_1 y_t + \tilde{a}'_h z_t + \tilde{b}'_h y_t \right),$$

where  $(\tilde{a}'_h, \tilde{b}'_h) = a'(\omega_{H-h+1} + (\tilde{a}'_{h-1}, \tilde{b}'_{h-1}))$ ,  $\tilde{a}_0 = 0$  and  $\tilde{b}_0 = 0$ .

Taking  $a_h = a_1 - \tilde{a}_h$ ,  $b_h = b_1 - \tilde{b}_h$ , with  $(a'_h, b'_h) = (a'_1, b'_1) - a'(\omega_{H-h+1} - (a'_{h-1} - a'_1, b'_{h-1} - b'_1)')$ , we get  $B(t, h) = \exp(-a'_h z_t - b'_h y_t)$ .

## A.6. Proof of Proposition 5

From Proposition 4, we have:

$$\begin{aligned} B_n^D(t, h) &= \exp(-a'_1 z_t - b'_1 y_t) E_t^{\mathbb{Q}} \left( -a'_1 z_{t+1} - b'_1 y_{t+1} - \alpha'_n z_{t+1} - \beta'_n y_{t+1} - \gamma'_n x_{n,t+1} - \dots \right. \\ &\quad \left. - a'_1 z_{t+h-1} - b'_1 y_{t+h-1} - \alpha'_n z_{t+h-1} - \beta'_n y_{t+h-1} - \gamma'_n x_{n,t+h-1} \right. \\ &\quad \left. - \alpha'_n z_{t+h} - \beta'_n y_{t+h} - \gamma'_n x_{n,t+h} \right). \end{aligned}$$

Using Lemma 2 with  $\omega_H = (-\alpha'_n, -\beta'_n, -\gamma'_n)$  and  $\omega_h = (-\alpha'_n - a'_1, -\beta'_n - b'_1, -\gamma'_n)$  for  $h = 1, \dots, H-1$ , we get:

$$B_n^D(t, h) = \exp \left( -a'_1 z_t - b'_1 y_t + \tilde{c}'_{n,h} z_t + \tilde{f}'_{n,h} y_t + \tilde{g}'_{n,h} x_{n,t} \right),$$

where  $(\tilde{c}'_{n,h}, \tilde{f}'_{n,h}, \tilde{g}'_{n,h}) = \tilde{a}'(\omega_{H-h+1} + (\tilde{c}'_{n,h-1}, \tilde{f}'_{n,h-1}, \tilde{g}'_{n,h-1}))$  and  $\tilde{c}_{n,0} = 0$ ,  $\tilde{f}_{n,0} = 0$  and  $\tilde{g}_{n,0} = 0$ .

Taking  $c_{n,h} = a_1 - \tilde{c}_{n,h}$ ,  $f_{n,h} = b_1 - \tilde{f}_{n,h}$  and  $g_{n,h} = -\tilde{g}_{n,h}$ , with  $(c'_{n,h}, f'_{n,h}, g'_{n,h}) = (a'_1, b'_1, 0) - \tilde{a}(\omega_{H-h+1} - (c'_{n,h-1} - a'_1, f'_{n,h-1} - b'_1, g'_{n,h-1})')$  and  $c_{n,0} = a_1$ ,  $f_{n,0} = b_1$ ,  $g_{n,0} = 0$ , we get  $B_n^D(t, h) = \exp(-c'_{n,h} z_t - f'_{n,h} y_t - g'_{n,h} x_{n,t})$ .

## A.7. Proof of Proposition 6 (Pricing of defaultable bonds with nonzero recovery rates)

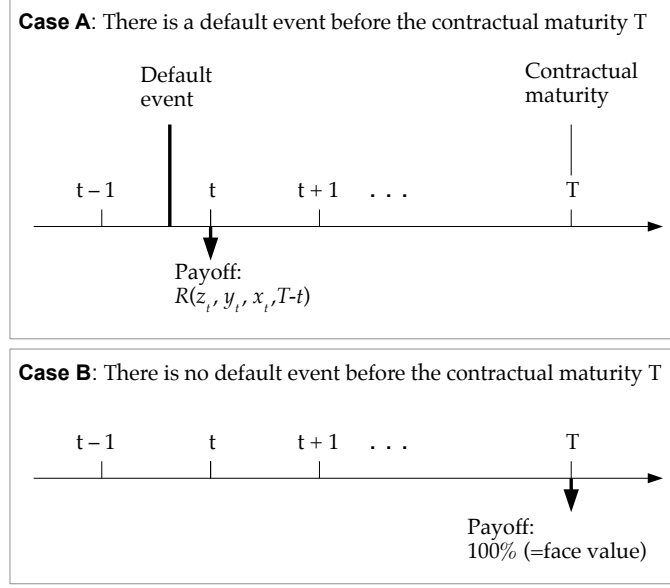
Section 4 gives quasi-explicit formulas for the pricing of bonds with zero recovery rates. In the current appendix, we present conditions under which one can derive formulas for nonzero-recovery-rate bond pricing. Figure 8 presents the payoff schedule considered here. As shown in this figure, if a debtor  $n$  defaults between  $t-1$  and  $t$  (with  $t < T$ , where  $T$  denotes the contractual maturity of a bond issued by this debtor), recovery is assumed to take place at time  $t$ . In addition, we assume that the recovery payoff—i.e. one minus the loss-given-default—depends on  $(z_t, y_t, x_t)$ . This recovery payoff is denoted by  $R_{n,t}^{T-t} := R(z_t, y_t, x_t, T-t)$ .

Let us consider the price  $B_n^{DR}(T-1, 1)$ , in period  $T-1$ , of a one-period nonzero-recovery-rate bond issued by a given debtor (before  $T-1$ ). We distinguish three cases:

1. The debtor had defaulted before  $T-2$ , then:  $B_n^{DR}(T-1, 1) = 0$ .
2. The debtor defaulted between  $T-2$  and  $T-1$ , then:  $B_n^{DR}(T-1, 1) = R_{n,T-1}^1$ .
3. The debtor has not defaulted before  $T-1$ , then:

$$\begin{aligned} B_n^{DR}(T-1, 1) &= \exp(-r_T) E^{\mathbb{Q}} \left[ \mathbb{I}_{\{d_{n,T}=0\}} + \mathbb{I}_{\{d_{n,T}=1\}} R_{n,T}^0 \mid \underline{z}_{T-1}, \underline{y}_{T-1}, \underline{x}_{n,T-1}, d_{n,T-1} = 0 \right] \\ &= \exp(-r_T) E^{\mathbb{Q}} \left[ E^{\mathbb{Q}} \left( \mathbb{I}_{\{d_{n,T}=0\}} + \mathbb{I}_{\{d_{n,T}=1\}} R_{n,T}^0 \mid \underline{z}_T, \underline{y}_T, \underline{x}_{n,T}, d_{n,T-1} = 0 \right) \right. \\ &\quad \left. \mid \underline{z}_{T-1}, \underline{y}_{T-1}, \underline{x}_{n,T-1}, d_{n,T-1} = 0 \right] \\ &= \exp(-r_T) E^{\mathbb{Q}} \left[ \exp(-\lambda_{n,T}) + (1 - \exp(-\lambda_{n,T})) R_{n,T}^0 \mid \underline{z}_{T-1}, \underline{y}_{T-1}, \underline{x}_{n,T-1}, d_{n,T-1} = 0 \right] \\ &= \exp(-r_T) E^{\mathbb{Q}} \left[ \exp(-\lambda_{n,T}) + (1 - \exp(-\lambda_{n,T})) R_{n,T}^0 \mid \underline{z}_{T-1}, \underline{y}_{T-1}, \underline{x}_{n,T-1} \right] \end{aligned}$$

Figure 8: **Payoffs stemming from a defaultable bond (issued before  $t - 1$ )**



and, defining the random variable  $\tilde{\lambda}_{n,T}^0$  by  $\exp(-\tilde{\lambda}_{n,T}^0) = \exp(-\lambda_{n,T}) + (1 - \exp(-\lambda_{n,T})) R_{n,T}^0$ , we have (still in case 3):

$$B_n^{DR}(T-1, 1) = E^{\mathbb{Q}} \left[ \exp(-r_T - \tilde{\lambda}_{n,T}^0) \mid \underline{z}_{T-1}, \underline{y}_{T-1}, \underline{x}_{n,T-1} \right].$$

Further, let us consider the price of the same bond in period  $T-2$ . Assuming that there was no default before  $T-2$ :

$$\begin{aligned} B_n^{DR}(T-2, 2) &= \exp(-r_{T-1}) \times \\ &E^{\mathbb{Q}} \left[ \mathbb{I}_{\{d_{n,T-1}=0\}} \left( E^{\mathbb{Q}} \left[ \exp(-r_T - \tilde{\lambda}_{n,T}^0) \mid \underline{z}_{T-1}, \underline{y}_{T-1}, \underline{x}_{n,T-1} \right] \right) \right. \\ &\quad \left. + \mathbb{I}_{\{d_{n,T-1}=1\}} R_{n,T-1}^1 \mid \underline{z}_{T-2}, \underline{y}_{T-2}, \underline{x}_{n,T-2}, d_{n,T-2} = 0 \right] \end{aligned} \quad (29)$$

Let us introduce a random variable  $\zeta_{n,T-1}^1$  that is defined through:

$$R_{n,T-1}^1 = \zeta_{n,T-1}^1 E^{\mathbb{Q}} \left[ \exp(-r_T - \tilde{\lambda}_{n,T}^0) \mid \underline{z}_{T-1}, \underline{y}_{T-1}, \underline{x}_{n,T-1} \right].$$

With this notation, Equation (29) reads:

$$\begin{aligned} B_n^{DR}(T-2, 2) &= E^{\mathbb{Q}} \left[ \exp(-r_{T-1} - r_T - \tilde{\lambda}_{n,T}^0) (\mathbb{I}_{\{d_{n,T-1}=0\}} + \zeta_{n,T-1}^1 \mathbb{I}_{\{d_{n,T-1}=1\}}) \right. \\ &\quad \left. \mid \underline{z}_{T-2}, \underline{y}_{T-2}, \underline{x}_{n,T-2}, d_{T-2} = 0 \right] \\ &= E^{\mathbb{Q}} \left[ E^{\mathbb{Q}} \left\{ \exp(-r_{T-1} - r_T - \tilde{\lambda}_{n,T}^0) (\mathbb{I}_{\{d_{n,T-1}=0\}} + \zeta_{n,T-1}^1 \mathbb{I}_{\{d_{n,T-1}=1\}}) \right. \right. \\ &\quad \left. \left. \mid \underline{z}_{T-1}, \underline{y}_{T-1}, \underline{x}_{n,T-1}, d_{n,T-2} = 0 \right\} \mid \underline{z}_{T-2}, \underline{y}_{T-2}, \underline{x}_{n,T-2}, d_{n,T-2} = 0 \right] \\ &= E^{\mathbb{Q}} \left[ \exp(-r_{T-1} - r_T - \tilde{\lambda}_{n,T}^0) (\exp(-\lambda_{n,T-1}) + \zeta_{n,T-1}^1 (1 - \exp(-\lambda_{n,T-1}))) \right. \\ &\quad \left. \mid \underline{z}_{T-2}, \underline{y}_{T-2}, \underline{x}_{n,T-2} \right]. \end{aligned}$$

Then, defining the random variable  $\tilde{\lambda}_{n,T-1}^1$  by:

$$\exp(-\tilde{\lambda}_{n,T-1}^1) = \exp(-\lambda_{n,T-1}) + (1 - \exp(-\lambda_{n,T-1})) \zeta_{n,T-1}^1,$$



we get (conditionally on  $d_{n,T-2} = 0$ ):

$$B_n^{DR}(T-2, 2) = E^{\mathbb{Q}} \left[ \exp(-r_T - r_{T-1} - \tilde{\lambda}_{n,T}^0 - \tilde{\lambda}_{n,T-1}^1) \mid \underline{z}_{T-2}, \underline{y}_{T-2}, \underline{x}_{n,T-2} \right].$$

Applying this methodology recursively, it is easily seen that the price of a nonzero-recovery-rate defaultable bond of maturity  $h$  is given by (assuming no default before  $t$ , i.e. conditionally on  $d_{n,t} = 0$ ):

$$B_n^{DR}(t, h) = E^{\mathbb{Q}} \left[ \exp(-r_{t+h} - \dots - r_{t+1} - \tilde{\lambda}_{n,t+h}^0 - \dots - \tilde{\lambda}_{n,t+1}^{h-1}) \mid \underline{z}_t, \underline{y}_t, \underline{x}_{n,t} \right] \quad (30)$$

where the  $\tilde{\lambda}_{n,t+i}^{h-i}$ 's are defined recursively in  $i$  by the backward equation:

$$\exp(-\tilde{\lambda}_{n,t+i}^{h-i}) = \exp(-\lambda_{n,t+i}) + (1 - \exp(-\lambda_{n,t+i})) \zeta_{n,t+i}^{h-i}$$

where

$$\zeta_{n,t+i}^{h-i} = \begin{cases} \frac{R_{n,t+i}^{h-i}}{E^{\mathbb{Q}} \left[ \exp(-r_{t+h} - \dots - r_{t+i+1} - \tilde{\lambda}_{n,t+h}^0 - \dots - \tilde{\lambda}_{n,t+i+1}^{h-i-1}) \mid \underline{z}_{t+i}, \underline{y}_{t+i}, \underline{x}_{n,t+i} \right]} & \text{if } i < h \\ R_{t+h,0} & \text{if } i = h. \end{cases}$$

Looking at Equation (30), it is tempting to interpret the  $\tilde{\lambda}_{n,t+i}^{h-i}$ 's as “recovery-adjusted” hazard rates for debtor  $n$ . However, the dependency of these intensities on the maturity  $h$  of the considered bond is problematic. Indeed, by analogy with the standard default intensities  $\lambda_{n,t}$ , one would like to have, at each period, only one adjusted intensity by debtor (and not a collection of adjusted intensities associated with the different bonds that have been issued by the considered debtor). To that end, Duffie and Singleton (1999) [28] propose a “recovery of market value” assumption. Under this assumption, the variable  $R_{n,s}^m$ —that is, the recovery at time  $s$  of a bond with residual maturity  $m$ , in the event of default between  $s-1$  and  $s$ —is equal to the product of a factor *common to all maturities* with the survival-contingent market value at time  $s$ . In the same spirit, let us assume that the  $\zeta_{n,s}^m$ 's do no longer depend on  $m$ . Then, the  $\tilde{\lambda}_{n,s}^m$  do not depend on the maturity any longer and are simply given by:

$$\exp(-\tilde{\lambda}_{n,s}) = \exp(-\lambda_{n,s}) + (1 - \exp(-\lambda_{n,s})) \zeta_{n,s}.$$

Actually, this formulation is more general than the one considered by Duffie and Singleton (1999) when they expose a discrete-time motivation. Indeed, in the latter case, they assume that  $\zeta_{n,s}$  is known at time  $s-1$ , which is not necessarily the case in the framework described above.

## B. Kitagawa-Hamilton algorithm for partially-hidden Markov chains

In this appendix, we describe how to use the Hamilton's (1990) [42] algorithm within the estimation strategy presented in Section 6, when the Markov chain is partially observed. While the algorithm was originally presented in a model with fixed transition probabilities, it readily generalizes to processes in which transition probabilities depend on a vector of observed variables.<sup>23</sup>

Let us denote with  $\hat{y}_t$  the vector of observed variables  $(\tilde{y}_t', R_{1t}, z_{1t}')'$ . The Hamilton's algorithm consists in computing recursively the probabilities  $p(z_{2t} \mid \hat{y}_t)$ . As a by product, the algorithm provides the conditional densities  $f(\hat{y}_t \mid \hat{y}_{t-1})$ , which makes it possible to estimate the model parameters by maximization of the log-likelihood. The algorithm is based on the iterative implementation of the following steps (the input being  $p(z_{2t-1} \mid \hat{y}_{t-1})$ ):

<sup>23</sup>See e.g. Filardo (1994) [33] for implementation examples of Hamilton's algorithm in models with time-varying transition probabilities.

1. The joint probability  $p(z_{2t}, z_{2t-1} | \hat{y}_{t-1})$  is computed using:

$$p(z_{2t}, z_{2t-1} | \hat{y}_{t-1}) = p(z_{2t} | z_{2t-1}, \hat{y}_{t-1}) \times p(z_{2t-1} | \hat{y}_{t-1})$$

where the first term of the right-hand side is a sum of entries of the transition matrix  $\{\pi_{ij,t-1}\}$  and the second term is the input.

2. The joint conditional density  $f(\hat{y}_t, z_{2t}, z_{2t-1} | \hat{y}_{t-1})$  is then given by:

$$f(\hat{y}_t, z_{2t}, z_{2t-1} | \hat{y}_{t-1}) = f(\hat{y}_t | z_{2t}, z_{2t-1}, \hat{y}_{t-1}) \times p(z_{2t}, z_{2t-1} | \hat{y}_{t-1})$$

where

$$\begin{aligned} f(\hat{y}_t | z_{2t}, z_{2t-1}, \hat{y}_{t-1}) &= f(\tilde{y}_t, R_{1t}, z_{1t} | z_{2t}, z_{2t-1}, \hat{y}_{t-1}) \\ &= f(\tilde{y}_t, R_{1t} | z_{1t}, z_{2t}, z_{2t-1}, \hat{y}_{t-1}) \times p(z_{1t} | z_{2t}, z_{2t-1}, \hat{y}_{t-1}) \end{aligned}$$

with

$$p(z_{1t} | z_{2t}, z_{2t-1}, \hat{y}_{t-1}) = \frac{p(z_{1t}, z_{2t} | z_{2t-1}, \hat{y}_{t-1})}{p(z_{2t} | z_{2t-1}, \hat{y}_{t-1})}$$

and all the terms can be computed.

3. The conditional density  $f(\hat{y}_t | \hat{y}_{t-1})$  is given by:

$$f(\hat{y}_t | \hat{y}_{t-1}) = \sum_{z_{2,t}} \sum_{z_{2,t-1}} f(\hat{y}_t, z_{2t}, z_{2t-1} | \hat{y}_{t-1}).$$

4. The joint density  $p(z_{2t}, z_{2t-1} | \hat{y}_t)$  comes from:

$$p(z_{2t}, z_{2t-1} | \hat{y}_t) = \frac{f(\hat{y}_t, z_{2t}, z_{2t-1} | \hat{y}_{t-1})}{f(\hat{y}_t | \hat{y}_{t-1})}.$$

5. And eventually:

$$p(z_{2t} | \hat{y}_t) = \sum_{z_{2,t-1}} p(z_{2t}, z_{2t-1} | \hat{y}_t).$$

## C. About the eigenvectors of the rating-migration matrix $\Pi$

In this appendix, using the notations presented in Subsection 8.4, we outline some properties of matrices  $\Pi$  and  $V$ . For notational simplicity, we drop arguments and time subscripts associated with these matrices.

- As the sum of the entries of each line of  $\Pi$  is equal to 1, the vector  $[1 \ \cdots \ 1]'$  is an eigenvector of  $\Pi$  associated with the eigenvalue 1. Consequently, since this eigenvalue is supposed to be the last one appearing in  $\Psi$ , the last column of  $V$ —that collects the eigenvectors of  $\Pi$ —is proportional to  $[1 \ \cdots \ 1]'$ .
- The fact that default is an absorbing state implies that the last row of  $\Pi$  is  $[0 \ \cdots \ 0 \ 1]$ . Since we have  $\Pi V = V \Psi$ , it comes (considering the last line of this equation):

$$\forall j \quad V_{K,j} = V_{K,j} \exp(-\psi_j),$$

which implies:  $\forall j < K, V_{K,j} = 0$ .

- The two previous points imply that the matrix  $V$  admits the following form:

$$V = \begin{bmatrix} V_{1,1} & \cdots & V_{1,K-1} & \gamma \\ \vdots & \ddots & \vdots & \vdots \\ V_{K-1,1} & \cdots & V_{K-1,K-1} & \gamma \\ 0 & \cdots & 0 & \gamma \end{bmatrix}$$

Since  $VV^{-1} = I$ , we have (considering the last line and using the notation  $V_{i,j}^{-1}$  for the entry  $(i,j)$  of  $V^{-1}$ )

$$\begin{bmatrix} V_{K,1}^{-1} & \cdots & V_{K,K-1}^{-1} & V_{K,K}^{-1} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{\gamma} \end{bmatrix}$$

and, therefore, for  $i = 1, \dots, K$ , we have  $V_{i,K}V_{K,K}^{-1} = 1$ .

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