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State-Dependent Probability Distributions in Non Linear Rational Expectations Models*

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Résumé

Dans ce papier, nous proposons une méthode de résolution de modèles non linéaires à anticipations rationnelles dans lesquels les changements de régimes ou les chocs eux-même peuvent être "endogènes", c'est-à-dire suivre des distributions de probabilités dépendant de l'état de l'économie. Par une méthode de perturbation, nous trouvons des conditions de détermination, i.e. des conditions d'existence d'un unique équilibre stable. Nous montrons que ces conditions découlent directement des conditions correspondantes dans le modèle à changements de régimes exogènes. Bien que ces conditions soient difficiles à vérifier dans le cas général, nous donnons, dans le cas des modèles à changements de régimes purement tournés vers le futur, des conditions de détermination faciles à calculer et une approximation au premier ordre de la solution. Enfin, nous illustrons nos résultats avec un modèle de Fisher de détermination d'inflation dans lequel la règle de politique monétaire change entre les régimes selon une matrice de transition dépendant de l'état de l'économie.
Codes JEL : E32, E43

Mots clés :Méthodes de perturbations, politique monétaire, indétermination, changements de régimes, DSGE.

Abstract

In this paper, we provide solution methods for non-linear rational expectations models in which regime-switching or the shocks themselves may be "endogenous", i.e. follow state-dependent probability distributions. We use the perturbation approach to find determinacy conditions, i.e. conditions for the existence of a unique stable equilibrium. We show that these conditions directly follow from the corresponding conditions in the exogenous regime-switching model. Whereas these conditions are difficult to check in the general case, we provide for easily verifiable and sufficient determinacy conditions and first-order approximation of the solution for purely forward-looking models. Finally, we illustrate our results with a Fisherian model of inflation determination in which the monetary policy rule may change across regimes according to a state-dependent transition probability matrix.
JEL classification: E32, E43

Keywords: Perturbation methods, monetary policy, indeterminacy, regime switching, DSGE.

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1 Introduction

Modern macroeconomic analysis explains economic fluctuations by micro-founding models on the time-invariant behavior of economic agents. Such an assumption, however, appears to be both questionable as a matter of principle, and problematic from an empirical standpoint (Clarida et al., 2000). Recent papers (e.g., Svennson and Williams, 2009; Farmer et al., 2009b) have thus relaxed the assumption of time-invariant behaviors by allowing recurring shifts in structural parameters. Whereas these papers suppose state-invariant transition probabilities, there is *a priori* no reason to assume independence between switches and the state of the economy(Kim et al., 2008). The aim of this paper is to allow for state-dependent transition probabilities in an otherwise standard non-linear rational expectations model.

We thus consider a class of non-linear discrete-time rational expectations models with both discrete jump process, describing regime switching, and continuous stochastic processes, the usual "shocks". Both processes may follow state-dependent probability distributions meaning that the distribution of shocks as well as the probabilities of regime switches may depend on the state of the economy. In this context, we find sufficient determinacy conditions, i.e. conditions insuring the existence of a unique stable solution, and provide the first order Taylor expansion of the solution. Our resolution strategy is based on perturbation approach as in Woodford (1986) and Jin and Judd (2002). Our findings are fourfold.

First, we prove that determinacy conditions for non linear rational expectations models with state dependent probability distributions of shocks and endogenous regime switching is related to determinacy conditions for the underlying exogenous regime switching model without shocks.

Second, by applying this result to the case of a one-regime model, we extend Woodford (1986) results in a context of non-linear rational expectations model with state dependent distributions of shocks. If the linearized model without shock admits only one stable solution then there exists a unique stable solution for the original model in a neighborhood of the steady state.

Third, we solve forward-looking regime switching model. We provide determinacy conditions which are easily verifiable based on eigenvalues computation. We thus generalize Farmer et al. (2009a) to multivariate forward-looking regime switching model. The state dependence of transition probabilities does not modify determinacy conditions but can have first order implication for the solution of the model.

Finally, we apply our method to a Fisherian model of inflation determination in which the monetary policy rule may change across regimes according to a state-dependent jump process. We find similar conditions to Farmer et al. (2009a) which extends Davig and Leeper (2007). Finally, we simulate impulse response functions using the first order Taylor expansion of the solution. For plausible set of parameters, we find that the endogenous nature of regime switching can trigger significant impact to the level of inflation in each regime; on the other hand, the simulations based on the endogenous regime switching model can also differ from those of the exogenous regime switching due to the endogenous fluctuations of the transition probabilities.

Related Literature Recent papers have challenged the empirical validity of models based on time-invariant behavior of economic agents. For instance, several papers analyze the sharp decreases in output and inflation volatility around the mid 80s', the so-called "Great Moderation", by allowing for time-varying economic behavior. Among the competing sources of parameter changes, some papers have allowed for breaks in the variance of structural shocks (Sims and Zha, 2006; Justiniano and Primiceri, 2008; Fernández-Villaverde et al., 2010; Liu et al., 2010), others for shifts in the parameters of monetary policy rules (Clarida et al., 2000; Lubik and Schorfheide, 2004).

Within the context of forward looking economic agents, the possibility of future shifts

should alter the agents' current decision rules (Sims, 1982) through what Leeper and Zha (2003) call the *expectations formation effects*. Such shifts in expectations could alter the determinacy conditions. However, the literature is rather mute on these conditions.

The literature on solving rational expectations models with time-varying parameters is quite vast see e.g. the papers by Justiniano and Primiceri (2008) and Fernández-Villaverde et al. (2010) on continuous changes or the papers by Davig and Leeper (2007, 2008), Farmer et al. (2009b, 2010b), Cho (2009), Svennson and Williams (2009) on regime switching. Although these papers provide deep analysis of the expectations, most of them deal with exogenous law of motion, whereas it would be interesting to study **state-dependent shifts** for a normative as well as a forecasting purpose. Under this respect Davig and Leeper (2008) is a notable exception, as in this paper the authors allow for state-dependent monetary policy rule parameters. Davig and Leeper's approach relies on computational methods to solve rational expectations models which imply both a high computational cost and the lack of analytical results.

Other than Davig and Leeper (2008) very few papers deal with state-dependent transition probabilities. We have to look at econometric literature to find significant contributions on this subject. Following the seminal paper by Hamilton (1989), Filardo (1994) and Filardo and Gordon (1998) estimate Markov switching regressions with time-varying transition probabilities. More recently, Kim et al. (2008) propose a technique for estimating multivariate models with endogenous regime switching - transition probabilities depend on endogenous variables. However, these significant progresses cannot be replicated yet to estimate rational expectations model with endogenous regime switching.

The absence of papers dealing with endogenous regime switching or state-dependent probability distributions in general in rational expectations context certainly stems from the fact that seminal papers aiming atsolving rational expectations models (Blanchard and Kahn, 1980; Woodford, 1986, and so on) have developed a consistent framework only for exogenous stochastic process. Dealing with endogenous regime switching thus requires to redefine the concept of a solution and to provide an adequate theoretical framework.

Concerning the resolution of (exogenous) Markov switching models, the literature has mainly focused on linear Rational Expectations models. In this class of models, we can distinguish two main resolution techniques: the undetermined coefficient approach (Blake and Zampolli, 2006; Davig and Leeper, 2007; Svennson and Williams, 2009) and a direct approach (Farmer et al., 2010b). Our approach is closer to the undetermined coefficient approach since both methods are equivalent when models are linear.

The characterization of the full class of solutions (the existence, the uniqueness and the form of the solution) in the context of Markov switching models is a challenging task. Davig and Leeper (2007) propose a simple determinacy condition in the context of forward-looking Markov switching models. However, Farmer et al. (2010a) have casted doubts about their results. Since this controversy, most of the literature has turned to the Mean Square Stability concept (see Farmer et al., 2009b) following the influential book by Costa and R.Marques (2005). This definition of stability is however incompatible with perturbation approach[1] and hence do not provide the adequate stability concept for solving non-linear Markov Switching DSGE models with perturbation theory.

In fact, little attention has been paid to Markov switching non-linear rational expectations models. One notable exception is Foerster et al. (2011). In this paper, the authors propose a resolution technique aiming at solving this model by applying a perturbation approach. Davig and Doh (2008) also solves a non-linear Markov Switching DSGE model by linearizing it and then using undetermined coefficient approach. Nevertheless, none of these papers give explicit arguments to apply the Implicit Function Theorem and prove that there exists a unique "stable" solution.

[1]Applying Implicit Function Theorem requires Banach spaces

The remainder of the paper is organized as follows. Section 2 presents the class of model we study as well as some probabilistic backgrounds. Then, we provide our main theoretical results in Section 3. Finally, Section 4 illustrates our findings through an endogenous regime switching Fisherian model of inflation determination.

2 Models

This section presents the class of models we consider in this paper and precise some probabilistic background needed for our analysis.

Most of recent rational expectations macroeconomic models with regime switching can be reduced to the following system:

$$\mathbb{E}_t[f_{s_t}(z_{t+1}, z_t, z_{t-1}, \gamma v_t)] = 0. \tag{1}$$

z is a vector of endogenous variables evolving in a bounded closed set F of $\mathbb{R}^n$. v is a multi-dimensional stochastic process evolving in a bounded domain V of $\mathbb{R}^p$ and γ is a scalar evolving in $]-M, M[$. s_t denotes the regime of the economy at date t and can take values in $\{1, \cdots N\}$ where N is the number of possible regimes. For any $i \in \{1, \cdots N\}$, f_i is a regular function from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}^n$ and $\mathbb{E}_t$ is the expectation operator given the information available at time t, namely current and past shocks and regimes. In the literature, the stochastic process v is assumed to follow a **given** probability distribution function (in most cases, Gaussian distributions). In this case, we say that the shocks are exogenous. Similarly, in Markov switching rational expectations literature, the authors assume the regime switching to be purely exogenous. By contrast, we allow the distributions of regime switching and of the shocks to depend on the state of the economy. This departure constitutes the core of this paper.

Let us precise the exact probability measures of shocks and regimes we consider. We denote by $u_t \in \{1, \cdots, N\} \times V$ the concatenation of the current shock and regime: (s_t, v_t). $\mathcal{M}(U)$ is the set of measures on U and by $U^\infty = \{1, \cdots, N\}^\infty \times V^\infty$, the space of infinite sequences $u^t = (u_t, u_{t-1}, \cdots)$[2]. We denote by $\mathcal{B}$ the set of functions Φ on $\{1, \cdots, N\}^\infty \times V^\infty$ such that, for all $s \in \{1, \cdots, N\}$, the map $v \mapsto \Phi(s, v)$ is continuous and such that $(s, v) \mapsto \Phi(s, v)$ is bounded. We define Σ, the sigma field of U generated by the product of the singletons $\{i \in \{1, \cdots, N\}\}$ and the Borel set of V. We consider a fixed map μ_γ:

$$\begin{cases} \mu_\gamma & : & \mathcal{B} \times U^\infty & \rightarrow & \mathcal{M}(U) \\ & & (\phi, s^{t-1}, v^{t-1}) & \mapsto & \mu_\gamma(\cdot, \phi, u^{t-1}) \\ \mu_\gamma(s, v, \phi, u^{t-1}) & = & \sum_{i=1}^{N} h^i(v, \phi, u^{t-1}) p_\gamma^i(\phi, u^{t-1}) \delta_i(s) & & \end{cases} \tag{2}$$

This measure is a combination of Lebesgue-continuous measure, h^i, and mass-point measures. δ_i denotes the Dirac distribution in i. We suppose that p_γ^i and h^i for any $i \in \{1, \cdots, N\}$ are smooth. We focus on this particular class of measures as they behave conveniently and encompass a large class of economic models. We present some useful properties of this class of measures in appendix A. It is worth noticing that we allow the measure of v to vary across regimes as its probability measure depends on i in the sum. We assume that h^i satisfies:

$$\int_V h^i(v, \phi, u^{t-1}) dv = 1, \qquad \forall i \in \{1, \cdots, N\}, \quad \forall \phi \in \mathcal{B}, \quad \forall u^{t-1} \in U \tag{3}$$

We then recursively define the probability measure $\pi_\gamma(\phi)$ on Σ^∞, the infinite product of Σ:

[2] For more details about this formalism, the reader can refer to Woodford (1986).

$$\begin{array}{rcll} \pi_\gamma & : & \mathcal{B} & \rightarrow & \mathcal{M}(U^\infty) \\ & & \phi & \mapsto & \mu(\cdot,\phi) = \prod_{k=0}^{\infty} \mu_\gamma(u_{t-k}, \phi, u^{t-k-1}) \end{array} \tag{4}$$

We can now define a stationary rational expectation equilibrium of model (1):

Definition 1. *A stationary rational expectations equilibrium (s.r.e.e.) of model* (1) *is a* ***continuous*** *function* $\phi : U^\infty \rightarrow F$ *such that:*

1. $||\phi||_\infty = \sup_{U^\infty} \|\Phi(u^t)\| < \infty$
2. *If* u *is a* U *valued stochastic process associated with the probability measure* $\pi_\gamma(\phi)$ *Then* $E[f_{s_t}(\Phi(u^{t+1}), \Phi(u^t), \Phi(u^{t-1}), \gamma v_t)|u^t] = 0$

Furthermore, this solution is a steady state if Φ *is constant.*

When shocks are exogenous (h^i and p^i invariant), the usual assumptions is to assume "small" shocks (Woodford, 1986; Judd, 1996; Uhlig, 1999; Juillard, 2003, among others). In this case, by perturbing the deterministic model (the model without any shock) one can prove that, for shocks small enough, there is a unique s.r.e.e. of the non-linear model if there exists a unique s.r.e.e. of the linearized model around a steady state (solution of the deterministic model).

Such approach seems appropriate only in case of **small** regime switches. Foerster et al. (2011) propose a resolution procedure based on perturbation approach around a steady state to solve such regime switching models. However, in presence of **large** regime switches, this approach seems unappropriate. That is the reason why we define a continuum of probability measures, μ_γ, parameterized by the so-called scale parameter, $\gamma \in [0, 1]$. When γ is equal to 0, the continuous shocks disappear from the model (they hence become pure sunspot shocks) and we assume that the probabilities, p_0^i, do not depend on ϕ - exogenous regime switching. Consequently, $\gamma = 0$ corresponds to an exogenous regime switching model without shocks. γ thus measures simultaneously the size of the shocks and the degree of endogeneity of regime switching (the slope of the mass-point probabilities). We could have distinguished these two dimensions without any substantial modification.

Finally, Implicit Function Theorem (IFT) applied to this continuum of probability measures will give us the existence and the uniqueness of a s.r.e.e. when the continuous shocks are small enough and the regime switching is weakly endogenous[3].

3 Solving Rational Expectations Models with perturbation approach

In this section, we prove that there exists a unique s.r.e.e. of the model with small continuous shocks and weakly endogenous regime switching (i.e. γ small enough) if the underlying exogenous regime switching model admits a unique solution [Theorem 1]. We show that, in the absence of regime switching, this result extends Woodford (1986) result by allowing shocks' distribution to be state-dependent [Theorem 2]. Then, we give existence and uniqueness conditions for a non-linear endogenous regime switching model in a purely forward-looking context [Theorem 3]. As the latter conditions are uneasy to check, we also give more stringent but easy-to-check conditions [Proposition 1]. Finally, we illustrate this Proposition in the context of linear endogenous regime switching model and compute the first order Taylor expansion [Proposition 2].

[3]The exact definition of what "small" means in this context is largely ignored by the literature and will be the subject of further research.

3.1 General result

We begin with a general theorem which results from Implicit Function Theorem (IFT) in Banach Spaces (see Abraham et al., 1988, and section B.1 in the appendix). To apply this latter, we define an operator $\mathcal{N}$, whose zeros correspond to a *s.r.e.e*:

$$\mathcal{N}(\phi, \gamma) = \int_U f_{s_t}(\phi(uu^t), \phi(u^t), \phi(u^{t-1}), \gamma v_t)\mu_\gamma(u, \phi, u^t)du \tag{5}$$

As explained in section 2, we assume that the mass-point probabilities do not depend on ϕ when γ is equal to 0. Thus:

$$\mathcal{N}(\phi, 0) = \sum_{i=1}^{N} p_0^i(u^t) \int_V f_{s_t}(\phi(is^t, vv^t), \phi(u^t), \phi(u^{t-1}), 0)h^i(v, \phi, u^{t-1})dv \tag{6}$$

This operator corresponds to a model with purely exogenous regime switching and the shocks, v, are sunspot shocks. One may notice that transition probabilities can depend on past regime and shocks. This thus encompasses Markov Switching models. Besides, even if the shocks, v, are sunspots (do not appear in the model, f_{s_t}) their probability distribution functions depend on the equilibrium, ϕ.

Theorem 1. *We assume that there exists a continuous function $\phi_0 : \{1, \cdots, N\}^\infty \to F$ such that*

1. *ϕ_0 is a particular s.r.e.e. of the non-linear exogenous regime switching model without shocks ($\gamma = 0$): $\mathcal{N}(\phi_0, 0) = 0$*
2. *There exists a unique s.r.e.e of the linear exogenous rational expectations model*

$$\mathbb{E}_t[A(s^{t+1})x_{t+1}] + B(s^t)x_t + C(s^t)x_{t-1} = 0 \tag{7}$$

where $A(s^t)$, $B(s^t)$ and $C(s^t)$ depend on ϕ_0 and correspond to the linearization of model (5) *in $(\phi_0, 0)$.*

Then there exists γ_0 small enough such that, for any γ smaller than γ_0, there exists a unique s.r.e.e. *of model* (5) *around ϕ_0. Furthermore, the first Taylor expansion of the solution, $\phi(\gamma)$ in γ is given by:*

$$\forall u^t \in U, \phi(\gamma)(u^t) = \phi_0(s^t) + \gamma D_\phi \mathcal{N}(\phi_0, 0)^{-1} D_\gamma \mathcal{N}(\Phi_0, 0) + o(\gamma)$$

Proof. This theorem is a direct application of IFT in Banach Space. See appendix B.1 for more details. □

Basically, this result shows that the weakly endogenous regime switching model has the same properties than the exogenous one, and that the solutions are close. It derives very general conditions of determinacy from properties of the model with exogenous regime switching.

In the context of no regime switching and exogenous shocks, this theorem is similar to Woodford (1986), Theorem 2. The first hypothesis can usually be checked by hand and the second hypothesis coincides with Blanchard and Kahn conditions of the underlying linearized model. However, contrary to Woodford (1986), this theorem only gives sufficient conditions of determinacy but is mute on the reciprocal. In addition, the solution, $\phi(\gamma)$, is not necessarily recursive contrary to Jin and Judd (2002).

In most cases, these conditions are hardly verifiable. We thus derive two theorems from Theorem 1 for which conditions 1. and 2. can be verified by algebraic computations. Firstly, we show that in the absence of regime switching this theorem extends Theorem 2 by Woodford (1986) [Theorem 2]. Secondly, we prove that this Theorem allows for solving non-linear forward looking regime switching models [Theorem 3].

3.2 Case I: state-dependent probability distribution in the absence of regime switching

We first consider the model (5) in the absence of regime switching:

$$\mathbb{E}_t[f(z_{t+1}, z_t, z_{t-1}, \gamma v_t)] = 0 \quad (8)$$

where v^t follows a continuous law, $h(v, \phi, v^{t-1})$, $\mu_\gamma = h$. We assume that h is Lebesgue-continuous and C^1 according to its second component.

Let us assume that there exists a *steady state* of the model 8 when $\gamma = 0$. We denote it by $\bar{z}$ and will call it the *deterministic steady state*. Thus, $\bar{z}$ satisfies $f(\bar{z}, \bar{z}, \bar{z}, 0) = 0$.

Theorem 2. *If the Blanchard and Kahn conditions for the linearized model in $\bar{z}$ are satisfied, then, there exists $\gamma_0 > 0$ such that for γ smaller than γ_0, the model* (8) *has a unique* s.r.e.e. *Furthermore, the first order expansion of this solution coincides exactly with the solution of the linearized model.*

Proof. The proof is quite similar to Woodford(1986,Theorem 2). It consists of an application of Theorem 1 to continuous shocks around a deterministic steady state: $\phi_0 = \bar{z}$. See appendix B.3 for the detailed proof. □

This theorem generalizes Woodford(1986,Theorem 2) to the case of shocks with state-dependent probability distributions. In this context, what we call the Blanchard and Kahn conditions is the fact that the number of explosive eigenvalues of the linearized model is exactly equal to the number of endogenous variables (plus a rank condition). Amazingly, neither the underlying linearized model nor the first order Taylor expansion of the solution changes compared to the exogenous case.

3.3 Case II: endogenous regime switching in a forward-looking environment

Let turn to the regime switching model. We consider the following purely forward-looking model.

$$\mathbb{E}_t[f_{s_t}(z_{t+1}, z_t, \gamma v_t)] = 0. \quad (9)$$

We assume that the transition probability from regime i to regime j only depends on the past value of endogenous variables, $\phi(u^{t-1})$. Thus, we assume that there exists a function p_{ij} mapping $F\times]-M, M[$ to $[0, 1]$ such that:

$$\forall u^t \in U^\infty, p^j_\gamma(\phi, (is^{t-1}, v^t)) = p_{ij}(\phi(u^t), \gamma)$$

Furthermore, we assume that the probabilities p_{ij} are smooth (C^1) and constant for $\gamma = 0$ ($p_{ij}(., 0) = \bar{p}_{ij}$). We can check that the implied measure, μ_γ fits all the needed properties described in Section 2.

As in the absence of regime switching, we assume that there exists a solution, ϕ_0 of the model when there is no shock. In addition, we suppose that this solution only depends on the current regime, i.e., $\phi_0(is^{t-1}) = \bar{z}_i$ where $(\bar{z}_1, \cdots, \bar{z}_N)$ is solution of the following equations, for any $i \in \{1, \cdots, N\}$:

$$\sum_{j=1}^{N} \bar{p}_{ij} f_i(\bar{z}_j, \bar{z}_i, 0) = 0$$

The existence of such equilibrium is "reasonable" as $(\bar{z}_1, \cdots, \bar{z}_N)$ is solution of a $N \times n$ system.

We define for $(i,j) \in \{1, \cdots, N\}^2$:

$$\beta_i = \sum_{j=1}^{N} \bar{p}_{ij} \partial_2 f_i(\bar{z}_j, \bar{z}_i, 0) \quad \texttt{and} \quad A_{ij} = \bar{p}_{ij} \partial_1 f_i(\bar{z}_j, \bar{z}_i, 0)$$

For convenience we assume that β_i is invertible for any $i \in \{1, \cdots, N\}$ [4]. We then can define the useful operator series $\mathcal{A}_p$:

$$\mathcal{A}_p \ : \phi \mapsto \left((s^t, v^t) \mapsto \sum_{s_2, \cdots, s_p} (-A_{s_t s_1}) \beta_{s_1}^{-1} \cdots (-A_{s_{p-1} s_p}) \beta_{s_p}^{-1} F_{s_t} \cdots F_{s_2} F_{s_p} \phi \right) \tag{10}$$

Where by convention, $\mathcal{A}_0 = \mathbf{1}$ and F_i denotes the expectation operator conditional to regime i [5]

Theorem 3. *If the series of operators $\sum_p \mathcal{A}_p$ is convergent, then there exists γ_0 small enough such that for any γ smaller than γ_0, the model* (9) *admits a unique* s.r.e.e., *$\phi(\gamma)$.*

Proof. See appendix B.4 for the proof. □

This result leads to two remarks. First, we can extend this result to solve models with "small" backward-looking component by introducing another scale parameter factoring in the backward-looking component. We however are not able to find explicit determinacy conditions for endogenous regime switching model with any backward-looking components. Yet, the convergence of $\sum \mathcal{A}_p$ is hard to check in practice. We thus give tighter but easy-to-check conditions ensuring the convergence of the series $\sum \mathcal{A}_p$.

We fix an operator norm, $|||.|||$, on $\mathcal{M}_n(\mathbb{R})$. We introduce the matrix S_p defined for $p > 1$ by:

$$S_p = \left(\sum_{(k_1, \cdots, k_{p-1}) \in \{1, \cdots, N\}^{p-1}} |||A_{ik_1} \beta_{k_1}^{-1} \cdots A_{k_{p-1}j} \beta_j^{-1}||| \right)_{ij} \tag{11}$$

and, by convention,

$$S_1 = \left(|||A_{ij} \beta_j^{-1}||| \right)_{ij}$$

Proposition 1. *If there exists an integer p such that all the eigenvalues of S_p lie inside the unit circle, then the series $\sum_{p=0}^{\infty} \mathcal{A}_p$ is absolutely convergent.*

Proof. This proof is based on the sub-multiplicative property of any operator norms. We develop the proof in Appendix C. □

This Proposition leads to multiple remarks. First of all, in the absence of regime switching the condition that S_1 has no explosive eigenvalue exactly corresponds to Blanchard and Kahn (1980) conditions. Secondly, if the model is univariate ($n = 1$), then checking the eigenvalues of S_1 is enough as $S_p = S_1^p$. Thirdly, the determinacy condition found by Farmer et al. (2009a) in the Fisherian model of inflation determination coincides with our condition when $p = 1$ (more details are provided in Section 4).

[4] This condition is the counterpart of the rank condition in standard DSGE model.

[5] F_i is defined by:

$$F_i : \phi \mapsto F_i \phi(s^t, v^t) = \int_V \phi_0(is^t, vv^t) h_i(v, \phi, u^{t-1}).$$

We give details on these operators in appendix B.4.

3.4 Case III: endogenous regime switching in a forward-looking and linear environment

When a forward-looking regime switching model is linear, one may solve it for any size of the shocks. Therefore, there is no necessity to assume small shocks. We thus present a refinement of Theorem 3 in case of linear model (f_i is linear for any $i \in \{1, \cdots N\}$). More explicitly, we consider the following model:

$$A_{s_t} E_t(x_{t+1}) + B_{s_t} x_t + \sigma C_{s_t} v_t = 0 \tag{12}$$

where the probabilities of transitions from regime i to regime j are $p_{ij}(\gamma, \phi(u^{t-1}))$. Furthermore, we assume that shocks, v_t, follow a first order Vectorial Auto-Regressive process:

$$v_{t+1} = \Lambda v_t + \mu_t$$

Where μ follows a centered standardized truncated gaussian (whose p.d.f. is h). In this special case, we can perform a simple perturbation approach assuming weakly endogenous probabilities around the exogenous regime switching model **with** shocks.

Proposition 2. *If $\sum \mathcal{A}_p$ is convergent, then, for γ small enough, the model* (12) *admits a unique* s.r.e.e.*, $\phi(\gamma)$, satisfying:*

$$\phi(\gamma)(s^t, v^t) = B_{s_t}^{-1} R_{s_t} v_t + \gamma B_{s_t}^{-1} \mathcal{R} \sum_{j}^{N} \int_V \partial_1 p_{s_t j}(0, R_{s_t} B_{s_t}^{-1}(\Lambda v_t + \mu))\mu h(\mu) d\mu + o(\gamma) \tag{13}$$

where R is a matrix given by (29) *in Appendix D.*

Proof. This Proposition follows from Proposition 1. Proof is given in Appendix D. □

The determinacy condition is exactly the same as Proposition 1. Equation (13) gives the first order Taylor expansion of the unique *s.r.e.e.* of the model (12). The first term, $B_{s_t}^{-1} R v_t$, is the exact solution of the exogenous regime switching model. The second term corresponds to the first order wedge introduced by the state-dependence of transition probabilities. It naturally depends on the sensitivity of the probabilities according to the endogenous variables. The integral can be either computed by hands in simple example or numerically approximated when the probabilities are too complex (e.g. not polynomial). We use this result in the application described in Section 4.

4 A Fisherian model of inflation determination

Following Taylor (1993), economists often simplify the monetary policy behavior through an invariant contingent rule. The monetary policy interest rate is then modeled as a weighted sum of the deviation between inflation and the central bank's inflation target, of an output-gap and of a residual - the so-called monetary policy shock. This gross description succeeds in explaining and analyzing monetary policy decisions. In this framework, some authors (Woodford, 2003, among others) prove that the existence and uniqueness of a stable equilibrium deeply depends on the ability of the central bank to react to inflation pressures. Precisely, in a wide range of New Keynesian models, the existence of a determinate rational expectations equilibrium is characterized by the *Taylor principle*, i.e. the ability of a central bank to adjust its interest rate more than one-for-one with inflation.

Taylor (1999), Clarida et al. (2000) and Lubik and Schorfheide (2004) attribute the change from a highly volatile regime in the 70s' to a low volatile regime in the mid 80s', the so-called Great Moderation, to a switch from a passive (less than one-for one reaction to inflation)

to an active (more than one-for one reaction to inflation) monetary policy regime. Thus it seems crucial to be able to model such a shift in monetary policy regime and to understand the implications in terms of determinacy.

Davig and Leeper (2007) provide necessary and sufficient conditions for determinacy in a Fisherian model of inflation determination as well as in a simple linearized New-Keynesian model both with Markov-Switching Taylor rule's parameters. They establish the counterpart of the *Taylor principle* in a Markov-Switching framework, they call it the *long run Taylor principle* -a combination between the transition probabilities and the central bank's reaction to inflation parameters has to be greater than one.

Farmer et al. (2010a) have casted doubts on Davig and Leeper's findings by providing a counter-example - a set of policy parameters satisfying the determinacy conditions proposed by Davig and Leeper but compatible with multiple bounded equilibria. In a companion paper, Farmer et al. (2009a) proves that determinacy conditions for the Fisherian model consist of a slightly modified version of the Davig and Leeper's *long run Taylor principle.*

Proposition 2 gives determinacy conditions as well as the first order Taylor expansion of the solution of the endogenous regime switching. As an illustration of endogenous regime switching, we analyze macroeconomic implications of having a "hawkish" central bank more concerned by limiting inflation than preventing deflation.

4.1 The model

Consider a nominal bond that costs 1 at date t and pays off $1 + i_t$ at date $t + 1$. Then, the asset pricing equation for this bond can be written in log form as:

$$i_t = E_t(\pi_{t+1}) + r_t \tag{14}$$

where r_t is the ex-ante equilibrium interest rate and evolves as

$$r_t = \rho r_{t-1} + v_t$$

where $\rho < 1$ and v_t is a zero-mean i.i.d bounded process. Monetary policy follows a simplified Taylor rule, adjusting the nominal interest rate in response to inflation, where the reaction to inflation evolves stochastically across regimes,

$$i_t = \alpha(s_t)\pi_t \tag{15}$$

where s_t is the realized monetary policy regime and takes two values $1, 2$. We assume that:

$$\alpha(s_1) = \alpha_1, \qquad \alpha(s_2) = \alpha_2$$

We use the formalism introduced in section 3.4 and assume that the switching process follows a Markov chain with transition probabilities $p_{ij} = p(s_t = j|s_{t-1} = i)$ depending on past inflation, π_{t-1}. To simplify the resolution and the exposition, we focus on probabilities satisfying:

$$p_{ij}(\pi_{t-1}) = \bar{p_{ij}} + \gamma(\lambda^1_{ij}\pi_{t-1} + \lambda^2_{ij}\pi^2_{t-1}) \tag{16}$$

Where λ^1_{ij} and λ^2_{ij} are two parameters reflecting the sensitivity of the probability, p_{ij}, to inflation and γ is the scale parameter. For consistency, $\sum_i \bar{p_{ij}} = 1$ and $\sum_i \lambda^1_{ij} = \sum_i \lambda^2_{ij} = 0$. Furthermore, we assume that γ is small enough to guarantee that the probabilities remain between 0 and 1 (this obviously requires that $\bar{p_{ij}} \in]\,0, 1\,[$).

As mentioned by Filardo (1994), endogenous regime switching as exemplified by equation (16) allows for state-dependent duration of each regime. In our example, if λ^1_{11} is positive and λ^2_{11} is zero then the average duration of regime 1 increases with the level of inflation.

4.2 The solution

We apply Propositions 1 and 2 to find determinacy conditions of the Fisherian model and a first order Taylor expansion of the solution.

Proposition 3. *If the policy parameters satisfy the following "modified"* Long Run Taylor Principle*:*

$$|\alpha_1|.|\alpha_2| + p_{22}(1 - |\alpha_1|) + p_{11}(1 - |\alpha_2|) > 1 \tag{17}$$

Then there exists a unique s.r.e.e. for γ small enough and the solution satisfies:

$$\pi_t = -\frac{r_t \Lambda_{s_t}}{\alpha_t} + \rho\gamma[a_{s_t} r_t^2 + b_{s_t} var(v)] + o(\gamma) \quad \textit{if} \quad \lambda_{ij}^2 = 0 \tag{18}$$

$$\pi_t = -\frac{r_t \Lambda_{s_t}}{\alpha_t} + \rho\gamma[c_{s_t} r_t^3 + d_{s_t} var(v) r_t] + o(\gamma) \quad \textit{if} \quad \lambda_{ij}^1 = 0 \tag{19}$$

Where a_{s_t}, b_{s_t}, c_{s_t}, d_{s_t} and Λ_{s_t} are constant only depending on the contemporaneous regime (see Appendix E for their expressions)

Proof. This Proposition is an application of Propositions 1 and 2. See appendix E for the complete proof. □

As emphasized by Theorem 3, the determinacy condition coincides with those of the exogenous Markov switching model. Condition (17) is similar to determinacy condition by Farmer et al. (2009a) but more stringent than Davig and Leeper (2007). Nevertheless, the interpretation of equation (17) is qualitatively similar to those of the *long-run Taylor principle* by Davig and Leeper (2007): there may exist a unique s.r.e.e. even if policy deviates from the Taylor principle "substantially for brief periods or modestly for prolonged periods".

Equation (18) (Equation (19)) gives the first order Taylor expansion of inflation with respect to the scale parameter when the probabilities are linear (quadratic resp.). The first term exactly coincides with the solution of the model when probabilities are constant.

When probabilities are linear, the second term of Equation (18) stems from the expectations of the volatility. When the real interest rate shock is i.i.d., the solution is exactly the solution of the exogenous model. The higher the variance of the real interest rate shock is, the larger the state-dependence of probabilities matters. Furthermore, the presence of r_t^2 is linked to the fact that the higher the shock today, the higher the expected volatility.

In the linear case, the wedge between the solution of the exogenous regime switching model and the solution of the endogenous one does not depend on the sign of the shock (Equation (18)) whereas this wedge is odd in the quadratic case and hence depends on the sign of the shock, r_t.

4.3 Numerical applications

Let us consider a central bank that can switch between an active monetary policy regime, let us say, $\alpha_2 = 2$ and a "neutral" regime in which monetary authority responds one to one to inflation, $\alpha_1 = 1$. Furthermore, we assume the central bank to be more likely to choose the active regime when inflation is high ($\lambda_{11}^1 < 0$).

Table 4.3 shows the values of the parameters. In this calibration, a quarterly inflation of 1% (−1%) in addition to the steady-state inflation corresponds to a 8% decrease (increase, resp.) in the probability to remain in regime 1. This situation may reflect a bias for fighting inflation rather than deflation. In this case, the first approximation of inflation is:

$$\text{When } s_t = 1, \qquad \pi_t = -0.44\% + 4.5 r_t - 33.3 r_t^2$$

$$\text{When } s_t = 2, \qquad \pi_t = -0.09\% + 1.4 r_t - 4.0 r_t^2$$

Parameter	Calibration
p_{11}	0.8
p_{22}	0.8
ρ	0.9
$\sqrt{var(v)}$	2%
λ_{11}^{1}	-8
λ_{22}^{1}	0
λ_{11}^{2}	0
λ_{22}^{2}	0

Table 1: Parameters calibration

As probabilities depend on the level of inflation, a positive and a negative shock will not lead to an identical response. To illustrate this asymmetry, we plot Impulse Response Functions to a positive and to a negative one-standard-deviation shock in Figures 1 and 2 in Appendix F.

The upper left graph shows the Impulse Response Function of a one-standard deviation shock, v_t, to the real interest rate, r_t, at date $t = 7$. The bottom left graph displays the response of inflation when the regime is fixed and endogenous (in thick lines) or fixed and exogenous (in dashed line). The fixed exogenous regime switching case coincides with Davig and Leeper responses, while the endogenous regime case brings new results. When the shock is positive - Figure 1- the inflation responses are smaller than the exogenous regime inflation responses in both regimes. This result stems from the negative contribution of the variance to the level of inflation. Nevertheless, the quadratic terms (in r_t^2) do not significantly matter in these observed differences. On the contrary, when the shock is negative, the responses in the endogenous case are larger (in absolute value). Altogether, these results are consistent with the fact that monetary authority is more likely to switch to the active monetary regime when inflation is large and thus has a more stabilizing policy when facing inflationary shocks rather than deflationary ones.

We finally plot (upper right) the share of the active monetary policy regime starting from the ergodic distribution of probabilities. We first notice that the ergodic distribution is not equally distributed among regimes as the variance term matters in the level of inflation even in the absence of shock. Second, as probabilities are state-dependent, the share of regime 2 is time-varying after a shock to the economy. Consequently, the average responses of inflation to a shock on the real interest rate is not the mean of the two fixed-regime responses. This can be seen in bottom right of figures 1 and 2 which display the average responses in the endogenous and the exogenous regime switching cases. Thus, endogenous fluctuations of probabilities generate significant effects in addition to the asymmetrical reaction mentioned before.

APPENDIX

A Measures μ_γ

We study stochastic process, $u = [s, v]$ with the following conditional probability distribution:

$$\mu_\gamma(s, v, \phi, u^{t-1}) = \sum_{i=1}^{N} h^i(v, \phi, u^{t-1}) p^i_\gamma(\phi, u^{t-1}) \delta_i(s)$$

Where we assume that:

- p^i_γ is C^1 according to its first argument and $\gamma \mapsto p^i_\gamma$ is C^1.
- For any $\phi \in \mathcal{B}$ and $u^{t-1} \in U^\infty$, $h^i(\cdot, \phi, u^{t-1})$ is an integrable function on V. h^i is also C^1 according to its second argument.
- For any $u^t \in U^\infty$, $\phi \mapsto h^i(v, \phi, u^{t-1})$ is C^1.

The measure $\mu_\gamma(\cdot, \phi, u^{t-1})$ can be seen as a continuous linear form on $\mathcal{B}$ and we write abusively $\int_U \mu_\gamma(u, \phi, u^{t-1})\Psi(u)du =< \Psi, \mu_\gamma >$ even if this measure is not Lebesgue-continuous in general.

Lemma 1. *1. For any $\Psi \in \mathcal{B}$, for any $\gamma \in [0, 1]$ and for any $u^{t-1} \in U^\infty$, $\phi \mapsto \int_U \Psi(u)\mu_\gamma(u, \phi, u^{t-1})du$ is C^1.*
We abusively denote by $\int_U \Psi(u)D_\phi\mu_\gamma(u, \phi, u^{t-1})(\hat{\phi})du$ its differential at ϕ applied to $\hat{\phi}$.

2. It exists $C > 0$, such that for any Ψ in $\mathcal{B}$ for any $\gamma \in [0, 1]$, for any $u^{t-1} \in U^\infty$, for any ϕ and $\hat{\phi}$ of norm equal to 1,

$$|\int_U \Psi(u)D_\phi\mu_\gamma(u, \phi, u^{t-1})(\hat{\phi})du| \leq C\|\Psi\|_\infty$$

3. For any $\Psi \in \mathcal{B}$, for any ϕ and for any $u^{t-1} \in U^\infty$, $\gamma \mapsto \int_U \Psi(u)\mu_\gamma(u, \phi, u^{t-1})du$ is $C^1([0, 1])$.

Proof. We first check that μ_γ satisfies 1. Fix $\Psi \in \mathcal{B}$, $\gamma \in [0, 1]$ and $u^{t-1} \in U^\infty$, we compute:

$$< \Psi, \mu_\gamma(u, \phi, u^{t-1}) >= \gamma \sum_{i=1}^{N} p^i_\gamma(\phi, u^{t-1}) \int_V h^i(v, \phi, u^{t-1})\Psi([i \quad v])dv$$

$\phi \mapsto< \Psi, \mu_\gamma(u, \phi, u^{t-1}) >$ is derivable; indeed, this function is the sum of the product of derivable function and an integral which is C^1 by dominated convergence theorem.
Then, we check that 2. is satisfied.

$$< \Psi, D_\phi\mu_\gamma(u, \phi, u^{t-1})(\hat{\phi}) >= \gamma \sum_{i=1}^{N} D_\phi p^i_\gamma(\phi, u^{t-1})(\hat{\phi}) \int_V D_\phi h^i(v, \phi, u^{t-1})(\hat{\phi})\Psi([i \quad v])dv$$

and

$$| < \Psi, D_\phi\mu_\gamma(u, \phi, u^{t-1})(\hat{\phi}) > | \leq (\sum_{i=1}^{N}\sum_{i=1}^{N} \sup |||D_\phi p^i_\gamma(., .)||| \sup ||||D_\phi h_\gamma(., ., .)||||)\|\Psi\|_\infty$$

Finally, we verify 3.

$$< \Psi, \mu_\gamma(u, \phi, u^{t-1}) >= \gamma \sum_{i=1}^{N} p^i_\gamma(\phi, u^{t-1}) \int_V h^i(v, \phi, u^{t-1})\Psi([i \quad v])dv$$

As we suppose that for any $i \in [1, N]$, $\gamma \mapsto p^i_\gamma$ is C^1 then $\gamma \mapsto< \Psi, \mu_\gamma(u, \phi, u^{t-1}) >$ is also C^1. □

These properties imply that the model as a whole is smooth enough to apply Implicit Function Theorem. Obviously, when measures are state-invariant, these two properties are immediate for any measures. We notice that point 2. can be interpreted as the fact that $D_\phi \mu_\gamma(u, \phi, u^{t-1})(\hat{\phi})$ is a distribution of order 0 uniformly bounded. Point 3. guarantees that the measures μ_γ are a C^1 path between μ_0 and μ_1 and hence we can apply perturbation method around $\gamma = 0$.

B Proof of Theorems 1,2 and 3

B.1 Proof of Theorem 1

In this part, we prove Theorem 1. The proof is a consequence of implicit function theorem applied to operator $\mathcal{N}$. First, we recall the Implicit Function Theorem(IFT).

Theorem 4. ***[Abraham et al. (1988)]*** *Let E, F, G be 3 Banach spaces, let $U \subset E, V \subset F$ be open and $f : U \times V \to G$ be C^r, $r \geq 1$. For some $x_0 \in U$, $y_0 \in V$ assume $D_y f(x_0, y_0) : F \to G$ is an isomorphism. Then there are neighborhoods U_0 of x_0 and W_0 of $f(x_0, y_0)$ and a unique C^r map $g : U_0 \times W_0 \to V$ such that, for all $(x, w) \in U_0 \times W_0$*

$$f(x, g(x, w)) = w$$

We denote by $\mathcal{S}$ the set of functions $\Phi : \{1, \cdots N\}^\infty \times V^\infty \mapsto \mathbb{R}^n$ such that:

- For all $s \in \{1, \cdots N\}^\infty$, $\Phi(s, \cdot)$ is continuous on V^∞.
- Φ is bounded on $\{1, \cdots N\}^\infty \times V^\infty$

Thus, we check that:

1. $\mathcal{B}$ with the norm $\|\|_\infty$, and $\mathbb{R}$ with $||$ are Banach spaces.
2. $\mathcal{N}$ is C^1 on $\mathcal{B}\times]-M, M[$.
3. Φ_0 satisfies $\mathcal{N}(\Phi_0, 0) = 0$.
4. $D_\Phi \mathcal{N}(\Phi_0, 0)$ is invertible.

The first point is immediate, $\mathcal{B}$ with the norm $\|\|_\infty$ is a Banach space as a product of Banach spaces. Point 2. results from the following lemma.

Lemma 2. *$(\Phi, \gamma) \mapsto \mathcal{N}(\Phi, \gamma)$ is C^1 for $\Phi \in \mathcal{B}$ and $\gamma \in]-M, M[$*

Proof. For any $\Phi \in \mathcal{B}$, the function $\gamma \mapsto \mathcal{N}(\Phi, \gamma)$ is C^1 by regularity of p^i_γ and f. For the differentiability in Φ, we check that $\Phi \mapsto \mathcal{N}(\Phi, \gamma)$ is differentiable, with continuous differential:

$$\mathcal{N}(\Phi, \gamma) = \sum_{i=1}^{N} p^i_\gamma(\Phi, u^{t-1}) \int_V f(\Phi(is^t, vv^t), \Phi(s^t, v^t), \Phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) h_i(v, \Phi) dv$$

$\Phi \mapsto \int_V f(\Phi(is^t, vv^t), \Phi(s^t, v^t), \Phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) h_i(v, \Phi) dv$ is differentiable by regularity of f, h_i, and Lebesgue's dominated convergence Theorem. It results from the differentiability of $\Phi \mapsto p^i_\gamma(\Phi, u^{t-1})$ that $\Phi \mapsto \mathcal{N}(\Phi, \gamma)$ is differentiable and moreover:

$$D_\Phi \mathcal{N}(\Phi, \gamma) H = \sum_{i=1}^{N} \partial_1 p^i_\gamma(\Phi, u^{t-1}) H(s^{t-1}, v^{t-1}) \int_V f(\Phi(is^t, vv^t), \Phi(s^t, v^t), \Phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) h_i(v, \Phi) dv$$

$$+\sum_{i=1}^{N} p_\gamma^i(\Phi, u^{t-1}) \int_V \partial_1 f(\Phi(is^t, vv^t), \Phi(s^t, v^t), \Phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) H(is^t, vv^t) h_i(v, \Phi) dv$$

$$+\sum_{i=1}^{N} p_\gamma^i(\Phi, u^{t-1}) \int_V \partial_2 f(\Phi(is^t, vv^t), \Phi(s^t, v^t), \Phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) H(s^t, v^t) h_i(v, \Phi) dv$$

$$+\sum_{i=1}^{N} p_\gamma^i(\Phi, u^{t-1}) \int_V \partial_3 f(\Phi(is^t, vv^t), \Phi(s^t, v^t), \Phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) H(s^{t-1}, v^{t-1}) h_i(v, \Phi) dv$$

$$+\sum_{i=1}^{N} p_\gamma^i(\Phi, u^{t-1}) \int_V f(\Phi(is^t, vv^t), \Phi(s^t, v^t), \Phi(s^{t-1}, v^{t-1}), s_t, \gamma v_t) H(s^{t-1}, v^{t-1}) \partial_2 h_i(v, \Phi) H(s^{t-1}, v^{t-1}) dv$$

Thus, $(\Phi, \gamma) \mapsto D_\Phi \mathcal{N}(\Phi, \gamma)H$ is continuous on $\mathcal{B}\times]-M, M[$. □

Points 3. and 4. result from assumptions of Theorem 1. We end the section with the following result, showing that the regularity of μ_γ implies that the differential of the operator behaves as the differential of an operator where the probabilities are exogenous.

Lemma 3. *Under assumptions of Theorem 1, the differential $D_\phi\mathcal{N}(\phi_0, 0)$ satisfies:*

$$\begin{aligned}D_\phi\mathcal{N}(\phi_0,0)H =& < \partial_1 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(\cdot S^t, \cdot v^t), \mu_0(\cdot, \phi_0, u^{t-1}) > + \\ & < \partial_2 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(s^t, v^t), \mu_0(\cdot, \phi_0, u^{t-1}) > + \\ & < \partial_3 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(s^{t-1}, v^{t-1}), \mu_0(\cdot, \phi_0, u^{t-1}) >\end{aligned}$$

Proof. We compute:

$$\begin{aligned}D_\phi\mathcal{N}(\phi_0,0)H =& < \partial_1 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(\cdot S^t, \cdot v^t), \mu_0(\cdot, \phi_0, u^{t-1}) > + \\ & < \partial_2 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(s^t, v^t), \mu_0(\cdot, \phi_0, u^{t-1}) > + \\ & < \partial_3 f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0) H(s^{t-1}, v^{t-1}), \mu_0(\cdot, \phi_0, u^{t-1}) > \\ & + < f(\phi_0(\cdot s^t, \cdot v^t), \phi_0(s^t, v^t), \phi_0(s^{t-1}, v^{t-1}), 0), D_\phi \mu_0(u, \phi_0, u^{t-1}) H(s^{t-1}, v^{t-1}) >\end{aligned}$$

Since ϕ_0 does not depend on v, the last term is zero. □

B.2 Notations

We introduce some notations, useful for the following. Fix a function $\phi_0 \in \mathcal{B}$, we define the operators F_i, for $i \in \{1, \cdots N\}$ and L on $\mathcal{B}$.

$$F_i : H \mapsto ((s^t, v^t) \mapsto \int_V H(is^t, vv^t) h_i(v, \phi_0, s^t, v^t) dv)$$

$$L : H \mapsto ((s^t, v^t) \mapsto H(s^{t-1}, v^{t-1})$$

Equation (3) imply that F_i and L have the following straightforward properties.

1. $F_i L = \mathbf{1}$
2. $|||F_i||| = 1$ and $|||L||| = 1$

Point 1. and point 2. are classical results in theory of operators on sequences. The first result is obtained by straightforward computation. The second follows from the fact that:

$$\forall H \in \mathcal{B}, \qquad \|LH\| = \|H\|, \qquad \|F_i H\| \leq \|H\|$$

and the last inequality is an equality if H is constant.

B.3 Proof of Theorem 2

This part is devoted to the proof of Theorem 2. We show that $\mathcal{N}$ satisfies points 1. and 2. of Theorem 1. Here, there is no discrete part, thus we omit the dependence in s_t and the indexation in i. The function ϕ_0 is the constant $\phi_0(v^t) = \bar{z}$. By construction, Φ_0 satisfies:

$$\mathcal{N}(\phi_0, 0) = f(\bar{z}, \bar{z}, \bar{z}, 0) = 0$$

We introduce the operator F associated to ϕ_0. Due to Lemma 2, $\mathcal{N}$ is differentiable and according to Lemma 3, we compute:

$$D_\Phi \mathcal{N}(\Phi_0, 0)h = \partial_1 f(\bar{z}, \bar{z}, \bar{z}, 0)Fh + \partial_2 f(\bar{z}, \bar{z}, \bar{z}, 0)h + \partial_3 f(\bar{z}, \bar{z}, \bar{z}, 0)Lh$$

Thus, following Woodford (1986) and Klein (2000), we will show that this operator is invertible when BK conditions are satisfied. Assume that

$$\partial_1 f(\bar{z}, \bar{z}, \bar{z}, 0)Fh + \partial_2 f(\bar{z}, \bar{z}, \bar{z}, 0)h + \partial_3 f(\bar{z}, \bar{z}, \bar{z}, 0)Lh = \Psi \tag{20}$$

Then,

$$\underbrace{\begin{pmatrix} g_2 & g_1 \\ I_n & 0 \end{pmatrix}}_{A} \begin{pmatrix} H \\ FH \end{pmatrix} = \underbrace{\begin{pmatrix} -g_3 & 0 \\ 0 & I_n \end{pmatrix}}_{B} L \begin{pmatrix} H \\ FH \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Psi$$

Fix $(s^t, v^t) \in \{1, \cdots, N\}^\infty \times V^\infty$, defining $z_t = H(s^t, v^t)$ and $z_{t+1} = FH(s^t, v^t)$, and $g_t = \Psi(s^t, v^t)$, we have to find bounded processes z_t such that:

$$\underbrace{\begin{pmatrix} g_2 & g_1 \\ I_n & 0 \end{pmatrix}}_{A} \begin{pmatrix} z_t \\ z_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} -g_3 & 0 \\ 0 & I_n \end{pmatrix}}_{B} \begin{pmatrix} z_{t-1} \\ z_t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} g_t \tag{21}$$

We see that as in Woodford (1986), the problem of invertibility leads to a question of existence and uniqueness of stationary solutions for linear models[6]. To deal with models where g_1 is not invertible, we generalize the approach of Woodford (1986) and follow Klein (2000), this leads to the following result.

Lemma 4. *Assume that Blanchard and Kahn conditions are satisfied for the linearized model (21), then $D_\Phi \mathcal{N}(\Phi_0, 0)$ is invertible and*

$$D_\Phi \mathcal{N}(\Phi_0, 0)^{-1} = (1 + Z_{22}^{-1} Z_{21} L)^{-1} Z_{22}^{-1} (1 - S_{22}^{-1} T_{22} P)^{-1} S_{22}^{-1} Q'_{12}$$

Proof. We use real generalized Schur decomposition on the pencil (A, B). There exist unitary matrices Q and Z, quasi triangular matrices T and S such that:

$$A = QTZ \quad \text{and} \quad B = QSZ$$

Furthermore, we rank the generalized eigenvalues such that $|T_{ii}| > |S_{ii}|$ for $i \in [1, n]$ and $|S_{ii}| > |T_{ii}|$ for $i \in [n+1, 2n]$ which is possible if and only if the number of explosive generalized eigenvalues is n (Blanchard and Kahn (1980), Klein (2000).

Considering,

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} z_t \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} z_{t-1} \\ z_t \end{pmatrix} + \begin{pmatrix} Q'_{11} & Q'_{21} \\ Q'_{12} & Q'_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} g_t \tag{22}$$

By using the last n rows and excluding divergent trajectories for $\hat{Z}_t$, we find:

[6] Woodford (1986, 2003) checks that the inverse of the derivative is continuous, but this is unnecessary due to the Banach isomorphism theorem

$$Z_{22}z_t = -Z_{21}z_{t-1} + \sum_{k=0}^{\infty}(S_{22}^{-1}T_{22})^k S_{22}^{-1} \begin{pmatrix} Q'_{11} \\ Q'_{12} \end{pmatrix} E_t(g_{t+k}).$$

If, in addition, a rank condition is verified (i.e. Z_{22} is of full rank), then Z_{22} is invertible and the solution is :

$$z_t = -Z_{22}^{-1}Z_{21}z_{t-1} + Z_{22}^{-1}\sum_{k=0}^{\infty}(S_{22}^{-1}T_{22})^k S_{22}^{-1}Q'_{12}E_t(g_{t+k}). \quad (23)$$

Using (23), we have that:

$$h(u^t) = -Z_{22}^{-1}Z_{21}(Lh)(u^t) + Z_{22}^{-1}\sum_{k=0}^{\infty}(S_{22}^{-1}T_{22})^k S_{22}^{-1}Q'_{12}P^k\Psi(u^t)$$

Thus,

$$h = (1 + Z_{22}^{-1}Z_{21}L)^{-1}Z_{22}^{-1}(1 - S_{22}^{-1}T_{22}P)^{-1}S_{22}^{-1}Q'_{12}\Psi$$

This ends the proof of the Lemma. □

Theorem 1 implies that there exists a small enough γ_0 and a unique bounded Φ such that:

$$\forall|\gamma| < \gamma_0, \quad \mathcal{N}(\Phi(\gamma),\gamma) = 0 \qquad \Phi(0) = \Phi_0$$

It proves that, for $|\gamma| < \gamma_0$, there exists a unique (Φ, π) satisfying model (8), Φ is given by Implicit Function Theorem and π is defined by:

$$\pi(u^t) = \tilde{p}(u_t, \Phi(u^{t-1}))$$

B.4 Proof of Theorem 3

In this section, we prove Theorem 3. We compute:

$$\mathcal{N}(\phi,0)(s^t,v^t) = \sum_{j=1}^{N}\bar{p}_{s_tj}\int_V f_{s_t}(\phi(js^t,vv^t),\phi(s^t,v^t),0)h_{s_t}(v,\phi,u^t)dv$$

We define the function ϕ_0 such that:

$$\forall i \in \{1,\cdots N\}, \qquad \phi_0(s^t,v^t) = \bar{z}_{s_t}$$

According to assumption 3., the function ϕ_0 satisfies :

$$\mathcal{N}(\phi_0,0) = 0$$

We compute now $D_\phi\mathcal{N}(\phi_0,0)$, using Lemma 3, we have:

$$D_\phi\mathcal{N}(\phi_0,0)h(s^t) = \sum_{j=1}^{N}\bar{p}_{s_tj}\partial_1 f_{s_t}(\bar{z}_j,\bar{z}_{s_t},0)F_jh$$

$$+(\sum_{j=1}^{N}\bar{p}_{s_tj}\partial_2 f_{s_t}(\bar{z}_j,\bar{z}_{s_t},0))h$$

Introducing A_{ij} and β_i, we get that:

$$D_\phi\mathcal{N}(\phi_0,0)h = \sum_{j=1}^{N}A_{s_tj}F_j + \beta_{s_t}F_{s_t}$$

Let Ψ a function in $\mathcal{B}(U^\infty)$ and consider the equation $D_\phi\mathcal{N}(\phi_0,0)h = \Psi$. Then, for all $i \in \{1,\cdots N\}$,

$$D_\phi\mathcal{N}(\phi_0,0)h = \sum_{j=1}^{N} A_{s_t j} F_j h + \beta_{s_t} F_{s_t} h = \Psi$$

This relation implies that for any $P \geq 2$

$$\begin{aligned} h &= \beta_{s_t}^{-1}\Psi - \beta_{s_t}^{-1}\sum_{p=2}^{P}(-1)^p \sum_{\{s_2,\cdots,s_p\}\in\{1,\cdots,N\}^{p-1}} A_{s_t s_2}\beta_{s_2}^{-1}\cdots A_{s_{p-1}s_p}\beta_{s_p}^{-1}F_{s_p}F_{s_{p-1}}\cdots F_{s_2}\Psi \\ &+ (-1)^P\beta_{s_t}^{-1}\sum_{\{s_2,\cdots,s_{P+1}\}\in\{1,\cdots,N\}^P} A_{s_t s_2}\beta_{s_2}^{-1}\cdots A_{s_{p-1}s_p}\beta_{s_p}^{-1}F_{s_{P+1}}F_{s_P}L_{s_{P-1}}\cdots F_{s_2}h \quad (24) \end{aligned}$$

We then define the operator series of general term, A_p:

$$A_p \;: \phi \mapsto \left((s^t,v^t) \mapsto \sum_{s_2,\cdots,s_p} (-A_{s_t s_1})\beta_{s_1}^{-1}\cdots(-A_{s_{p-1}s_p})\beta_{s_p}^{-1}F_{s_p}\cdots F_{s_2}F_{s_t}\phi \right)$$

If $\sum A_p$ converges, then, the third member of equation (24) tends to 0 when p tends to ∞ and the second member converges in $\mathcal{B}(U^\infty)$. Thus, h is uniquely defined for any $\psi \in \mathcal{B}(U^\infty)$ by:

$$h(s^t,v^t) \;=\; \beta_{s_t}^{-1}F_{s_t}\Psi - \beta_{s_t}^{-1}\sum_{p=2}^{\infty}\mathcal{A}_p\Psi = \beta_{s_t}^{-1}\sum_{p=1}\mathcal{A}_p\Psi \quad (25)$$

This results proves that for any $\Psi \in \mathcal{B}(U^\infty)$, we have found a unique solution h such that:

$$D_\Phi\mathcal{N}(\Phi_0,0)h = \Psi$$

Thus, $D_\Phi\mathcal{N}(\Phi_0,0)$ is invertible and that

$$D_\Phi\mathcal{N}(\Phi_0,0)^{-1}\Psi = \beta_{s_t}^{-1}\sum_{p=1}\mathcal{A}_p\Psi \quad (26)$$

C Proof of Proposition 1

In this part, we consider the matrix S_p defined by

$$S_p = \left(\sum_{(k_1,\cdots,k_{p-1})\in\{1,\cdots,N\}^{p-1}} |||A_{ik_1}\beta_{k_1}^{-1}\cdots A_{k_{p-1}j}\beta_j^{-1}||| \right)_{ij}$$

Fix p such that the eigenvalues of S_p lie inside the unit circle ; we will show that $\sum\mathcal{A}_p$ is absolutely convergent. For any $(q,r) \in \mathbb{N}\times\{0,\cdots,p-1\}$, we use sub-multiplicativity of norm $|||\cdot|||$ and compute, for $n = pq+r$:

$$\sum_{\{s_2,\cdots,s_n\}\in\{1,\cdots,N\}^{n-1}} |||A_{ik_1}\beta_{k_1}^{-1}\cdots A_{k_{p-1}j}\beta_j^{-1}||| \leq$$

$$\sum_{\{s_2,\cdots,s_{pq}\}\in\{1,\cdots,N\}^{pq-1}} |||A_{is_2}\beta_{s_2}^{-1}\cdots A_{s_{p-1}s_p}\beta_{s_p}^{-1}|||\cdots|||A_{s_{p(q-1)}s_{p(q-1)+1}}\beta_{s_{p(q-1)}}^{-1}\cdots A_{s_{pq-1}s_{pq}}\beta_{s_{pq}}^{-1}|||\times$$

$$\sum_{\{s_2,\cdots,s_r\}\in\{1,\cdots,N\}^{r-1}} |||A_{s_{pq}s_2}\beta_{s_2}^{-1}\cdots A_{s_{r-1}s_r}\beta_{s_r}^{-1}|||$$

We find an upper bound for both terms of the previous inequality. Concerning the second term, there exists $C>0$,such that for any $r\in\{0,\cdots,p-1\}$,

$$\sup_{i\in\{1,\cdots,N\}} \sum_{\{s_2,\cdots,s_r\}\in\{1,\cdots,N\}^{r-1}} |||A_{s_i s_2}\beta_{s_2}^{-1}\cdot A_{s_{r-1}s_r}\beta_{s_r}^{-1}||| < C \tag{27}$$

We rewrite the first term as:

$$\sum_{\{s_2,\cdots,s_{pq}\}\in\{1,\cdots,N\}^{pq-1}} |||A_{is_2}\beta_{s_2}^{-1}\cdots A_{s_{p-1}s_p}\beta_{s_p}^{-1}|||\cdots|||A_{s_{p(q-1)}s_{p(q-1)+1}}\beta_{s_{p(q-1)}}^{-1}\cdots A_{s_{pq-1}s_{pq}}\beta_{s_{pq}}^{-1}|||$$

$$= \sum_{s_{pq}\in\{1,\cdots,N\}} (S_p^q)_{is_{pq}} \tag{28}$$

Combining (27) and (C) leads to

$$|||\mathcal{A}_n||| < \sup_i \sum_{j=1}^{N} (S_p^q)_{ij}$$

[7] Then, denoting by $[x]$ the ? part of a real number x, we obtain :

$$|||\mathcal{A}_n||| \le C|||(S_p)^{\left[\frac{n}{p}\right]}|||_\infty$$

Since all the eigenvalues of S_p lie inside the unit circle, due to the Gelfand's Theorem, for any matrix norm,

$$\lim_{q\to\infty} |||S_p^q|||^{1/q} = \rho < 1$$

This implies that $\lim_{n\to+\infty} |||S_p^{\left[\frac{n}{p}\right]}|||^{1/n} = \sqrt[p]{\rho} < 1$. Finally, using the Cauchy rule, the series $\sum |||(S_p)^{\left[\frac{n}{p}\right]}|||_\infty$ is convergent and thus $\sum \mathcal{A}_n$ is absolutely convergent.

D Proof of Proposition 2

In this section, we prove Proposition 2. The proof relies on a refinement of the method used in Theorem 3. We first compute the solution when $\gamma=0$. Since the model is completely linear, the solution is defined for any γ. Then we solve the model for a small γ by perturbation around the model with exogenous probabilities.

We consider the following model:

$$A_{s_t}E_t(x_{t+1}) + B_{s_t}x_t + \sigma C_{s_t}v_t = 0$$

We assume that for any $i\in\{1,\cdots,N\}$, the matrices B_i are invertible. In this case, the operators $\mathcal{A}_p$ defined in equation (10) satisfy, for $p>0$:

$$\mathcal{A}_p \;: \Phi \mapsto ((s^t,v^t) \mapsto \sum_{s_2,\cdots,s_p} (-\bar{p}_{s_t s_2}A_{s_t})B_{s_t}^{-1}\cdots(-\bar{p}_{s_{p-1}s_p}A_{s_p})B_{s_p}^{-1}F_{s_t}\cdots F_{s_2}F_{s_p}\Phi)$$

[7] We recall that for a matrix $M\in\mathcal{M}_N(\mathbb{R})$, $\quad |||M|||_\infty = \sup_{i\in\{1,\cdots,n\}} \sum_{j=1}^{N} |M_{ij}|$

Defining $\Phi_0(s^t, v^t) = \sigma C_{s_t} v_t$ and using the computations in section B.4, we get that the solution of the model is given by :

$$\Phi(s^t, v^t) = \sum_{p=1}^{\infty} \mathcal{A}_p \Phi_0$$

First, we compute

$$\mathcal{A}_p \Phi_0 = \sigma \sum_{s_2, \cdots, s_p} (-\bar{p}_{s_t s_2} A_{s_t}) B_{s_t}^{-1} \cdots (-\bar{p}_{s_{p-1} s_p} A_{s_p}) B_{s_p}^{-1} C_{s_p} \Lambda^p v_t$$

Defining the matrix $P \in \mathcal{M}_{nN}(\mathbb{R})$ by blocks as:

$$P_{ij} = p_{ij} A_j (B_j)^{-1}$$

$$\mathcal{A}_p \Phi_0 = (P^p)_{s_t} \times C \times \Lambda^p$$

where $C = \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix}$. This leads to:

$$\sum_{p=1}^{\infty} \mathcal{A}_p \Phi_0 = \sigma (P^p)_{s_t} \times C \times \Lambda^p$$

This leads to the following result: Defining $R = \begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix}$ such that $\mathcal{R}\Phi_0(s^t, v^t) = R_{s_t} v_t$, then R satisfies:

$$\text{Vect}(R) = (I - (\Lambda \otimes P))^{-1} \text{Vect} \begin{pmatrix} C_1 \\ \vdots \\ C_N \end{pmatrix} \tag{29}$$

Consequently, the solution of this linear model when $\gamma = 0$ satisfies:

$$\phi(\sigma, 0)(s^t, v^t) = \sigma B_{s_t}^{-1} R_{s_t} v_t$$

We apply Theorem 1 and compute

$$D_\phi \mathcal{N}(\phi(\sigma, 0), \sigma, 0) h(s^t, v^t) = B_{s_t}^{-1} (\mathcal{R} h)(s^t, v^t)$$

and:

$$D_\phi \mathcal{N}(\phi(\sigma, 0), \sigma, 0) = \sigma \sum_{j=1}^{N} \partial_1 p_{s_t j}(0, \sigma B_{s_t}^{-1} R_{s_t} v) A_j B_j^{-1} R_j v h_{s_t}(v, v^t) dv$$

This implies that:

$$\phi(\sigma, \gamma)(s^t, v^t) = \sigma B_{s_t}^{-1} R_{s_t} v_t + \sigma R \sum_{j=1}^{N} \int_V \partial_1 p_{s_t j}(0, \sigma B_{s_t}^{-1} R_{s_t} (\Lambda v_t + \mu)) A_j B_j^{-1} R_j (\Lambda v_t + \mu) d\mu$$

E Proof of Proposition 3

E.1 Existence and uniqueness of a s.r.e.e.

This results from Proposition 1.

$$S_1 = \begin{bmatrix} \frac{p_{11}}{|\alpha_1|} & \frac{p_{12}}{|\alpha_2|} \\ \frac{p_{21}}{|\alpha_1|} & \frac{p_{22}}{|\alpha_2|} \end{bmatrix}$$

Here, $S_p = S_1^p$ and we only need to check that all eigenvalues of S_1 are smaller than 1. This condition exactly coincides with Farmer et al. (2009a) determinacy condition.

Furthermore, the eigenvalues of S_1 are smaller than one if and only if:

$$|\alpha_1|.|\alpha 2| + p_{22}(1 - |\alpha_1|) + p_{11}(1 - |\alpha_2|) > 1.$$

E.2 The solution when probabilities are exogenous

We compute the solution of the model

$$E_t \pi_{t+1} + r_t = \alpha_t \pi_t$$

$$r_t = \rho r_{t-1} + v_t$$

We have:

$$\pi_t = -\sum_{k=0}^{\infty} E_t \frac{r_{t+k}}{\prod_{j=0}^{k} \alpha_{t+j}}$$

By independency,

$$\pi_t = -\sum_{k=0}^{\infty} E_t(r_{t+k}) E_t \frac{1}{\prod_{j=0}^{k} \alpha_{t+j}}$$

$$\pi_t = -r_t \sum_{k=0}^{\infty} \rho^k E_t \frac{1}{\prod_{j=0}^{k} \alpha_{t+j}}$$

This implies that:

$$\pi_t = -\frac{r_t}{\alpha_t} \sum_{k=0}^{\infty} \rho^k E_t \frac{1}{\prod_{j=1}^{k} \alpha_{t+j}}$$

where $\prod_{j=1}^{0} \alpha_{t+j} = 1$ by convention.

Defining $\Lambda_t = \sum_{k=0}^{\infty} \rho^k E_t \frac{1}{\prod_{j=1}^{k} \alpha_{t+j}}$, we have:

$$\pi_t = -\frac{\Lambda_t r_t}{\alpha_t}$$

It remains to compute Λ_t. First, we compute:

$$E_t \frac{1}{\prod_{j=1}^{k} \alpha_{t+j}} = \sum_{(i_1,\cdots,i_k)\in\{1,2\}^k} \frac{p_{s_t i_1}}{\alpha_{i_1}} \frac{p_{i_1 i_2}}{\alpha_{i_2}} \cdots \frac{p_{i_{k-1} i_k}}{\alpha_{i_k}}$$

We define $P = \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix}$, and $A = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$. This could be rewritten as:

$$E_t \frac{1}{\prod_{j=1}^{k} \alpha_{t+j}} = (\tilde{P}^k)_{s_t} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $\tilde{P} = P * A^{-1}$ and $(\tilde{P}^k)_{s_t}$ stands for the s_t line of the matrix $\tilde{P}^k$. This leads to:

$$\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} = \sum_{k=0}^{\infty} \rho^k \tilde{P}^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, since $\|\rho\tilde{P}\| < 1$,

$$\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} = (I - \rho\tilde{P})^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In particular, if $\rho = 0$, $\Lambda_1 = \Lambda_2 = 1$.

E.3 Computation of $D_\gamma \mathcal{N}(\Phi_0, 0)$

The expression of $\mathcal{N}(\Phi, \gamma)$ leads to:

$$D_\gamma \mathcal{N}(\Phi_0, 0) = \rho \frac{\Lambda_{s_t}}{\alpha_{s_t}} (\frac{\Lambda_1}{\alpha_1} - \frac{\Lambda_2}{\alpha_2})[\lambda^1_{s_t 1} + \frac{\Lambda_{s_t}}{\alpha_{s_t}} \lambda^2_{s_t 1} r_t] r_t^2$$

We thus define $\beta_i = \frac{\Lambda_i}{\alpha_i}(\frac{\Lambda_1}{\alpha_1} - \frac{\Lambda_2}{\alpha_2})\lambda^1_{i1}$ and $\gamma_i = \frac{\Lambda_i^2}{\alpha_i^2}(\frac{\Lambda_1}{\alpha_1} - \frac{\Lambda_2}{\alpha_2})\lambda^2_{i1}$, to rewrite the differential as follows:

$$D_\gamma \mathcal{N}(\Phi_0, 0) = \rho(\beta_{s_t} r_t^2 + \gamma_{s_t} r_t^3)$$

E.4 Computation of $D_\Phi \mathcal{N}(\Phi_0, 0)^{-1} D_\gamma \mathcal{N}(\Phi_0, 0)$

Computing :

$$D_\Phi \mathcal{N}(\Phi_0, 0)^{-1} D_\gamma \mathcal{N}(\Phi_0, 0) = -\frac{r_t \Lambda_{s_t}}{\alpha_{s_t}} + \gamma\rho \sum_{k=0}^{\infty} E_t \frac{\beta_{t+k}}{\prod_{j=0}^k \alpha_{t+j}} r_{t+k}^2 + \gamma\rho \sum_{k=0}^{\infty} E_t \frac{\gamma_{t+k}}{\prod_{j=0}^k \alpha_{t+j}} r_{t+k}^3$$

As r_t is independent from the switching process,

$$D_\Phi \mathcal{N}(\Phi_0, 0)^{-1} D_\gamma \mathcal{N}(\Phi_0, 0) = -\frac{r_t \Lambda_{s_t}}{\alpha_{s_t}} + \gamma\rho \sum_{k=0}^{\infty} E_t \frac{\beta_{t+k}}{\prod_{j=0}^k \alpha_{t+j}} E_t r_{t+k}^2 + \gamma\rho \sum_{k=0}^{\infty} E_t \frac{\gamma_{t+k}}{\prod_{j=0}^k \alpha_{t+j}} E_t r_{t+k}^3$$

Let first remark that:

$$E_t \frac{\beta_{t+k}}{\prod_{j=0}^k \alpha_{t+j}} = \frac{1}{\alpha_{s_t}} \tilde{P}^k_{s_t} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

and

$$E_t \frac{\gamma_{t+k}}{\prod_{j=0}^k \alpha_{t+j}} = \frac{1}{\alpha_{s_t}} \tilde{P}^k_{s_t} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

Now we compute first and second raw moments of r_t:

$$E_t r_{t+k}^3 = 3var(v) E_t r_{t+k-1} + \rho^3 E_t r_{t+k-1}^3$$

$$E_t r_{t+k}^3 = 3var(v) r_t \sum_{i=1}^{k} (\rho^3)^{i-1} \rho^{k+1-i} + (\rho^3)^k r_t^3$$

$$E_t r_{t+k}^3 = 3\rho^k \frac{1 - (\rho^2)^k}{1 - \rho^2} var(v) r_t + (\rho^3)^k r_t^3$$

$$E_t r_{t+k}^3 = 3\frac{\rho^k}{1 - \rho^2} var(v) r_t + (\rho^3)^k [r_t^3 - \frac{var(v) r_t}{1 - \rho^2}]$$

Thus,

$$\gamma\rho\sum_{k=0}^{\infty}E_t\frac{\gamma_{t+k}}{\prod_{j=0}^{k}\alpha_{t+j}}E_tr_{t+k}^3=\gamma\frac{\rho}{\alpha_{s_t}}\frac{var(v)r_t}{1-\rho^2}\sum_{k=0}^{\infty}\rho^k\tilde{P}_{s_t}^k\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}+(r_t^3-\frac{var(v)r_t}{1-\rho^2})\sum_{k=0}^{\infty}\rho^{3k}\tilde{P}_{s_t}^k\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$$

$$\gamma\rho\sum_{k=0}^{\infty}E_t\frac{\gamma_{t+k}}{\prod_{j=0}^{k}\alpha_{t+j}}E_tr_{t+k}^3=\gamma\frac{\rho}{\alpha_{s_t}}\frac{var(v)r_t}{1-\rho^2}(I-\rho\tilde{P})^{-1}\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}+(r_t^3-\frac{var(v)r_t}{1-\rho^2})(I-\rho^3\tilde{P})^{-1}\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$$

Furthermore, we can show by recursion that:

$$E_tr_{t+k}^2=\frac{var(v)}{1-\rho^2}+\rho^{2k}[r_t^2-\frac{var(v)}{1-\rho_2}]$$

Consequently,

$$\gamma\rho\sum_{k=0}^{\infty}E_t\frac{\beta_{t+k}}{\prod_{j=0}^{k}\alpha_{t+j}}E_tr_{t+k}^2=\gamma\frac{\rho}{\alpha_{s_t}}\frac{var(v)}{1-\rho^2}\sum_{k=0}^{\infty}\tilde{P}_{s_t}^k\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}+(r_t^2-\frac{var(v)}{1-\rho^2})\sum_{k=0}^{\infty}\rho^{2k}\tilde{P}_{s_t}^k\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}$$

Thus,

$$\gamma\rho\sum_{k=0}^{\infty}(...)=\gamma\frac{\rho}{\alpha_{s_t}}\left[\frac{var(v)}{1-\rho^2}(I-\tilde{P})_{s_t}^{-1}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}\right.\quad(30)$$

$$\left.+\ (r_t^2-\frac{var(v)}{1-\rho^2})(I-\rho^2\tilde{P})_{s_t}^{-1}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}\right]\quad(31)$$

Finally by applying Proposition 2 we find:

$$\pi_t=-\frac{r_t\Lambda_{s_t}}{\alpha_{s_t}}+\rho\gamma(a_{s_t}r_t^2+b_{s_t}\frac{var(v)}{1-\rho^2}+c_{s_t}r_t^3+d_{s_t}\frac{var(v)}{1-\rho^2}r_t)+o(\gamma)$$

Where

$$\begin{bmatrix}a_1\\ a_2\end{bmatrix}=A^{-1}(I-\rho^2\tilde{P})^{-1}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}$$

$$\begin{bmatrix}b_1\\ b_2\end{bmatrix}=A^{-1}[(I-\tilde{P})^{-1}-(I-\rho^2\tilde{P})^{-1}]\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}$$

$$\begin{bmatrix}c_1\\ c_2\end{bmatrix}=A^{-1}(I-\rho^3\tilde{P})^{-1}\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$$

$$\begin{bmatrix}d_1\\ d_2\end{bmatrix}=A^{-1}[(I-\rho\tilde{P})^{-1}-(I-\rho^3\tilde{P})^{-1}]\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$$

F Figures

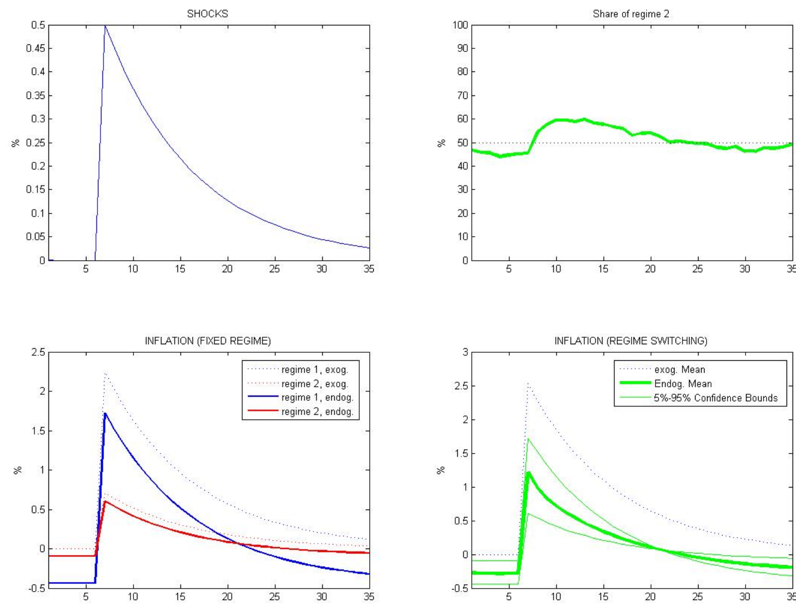

Figure 1: Impulse Response Function to a positive real interest rate shock

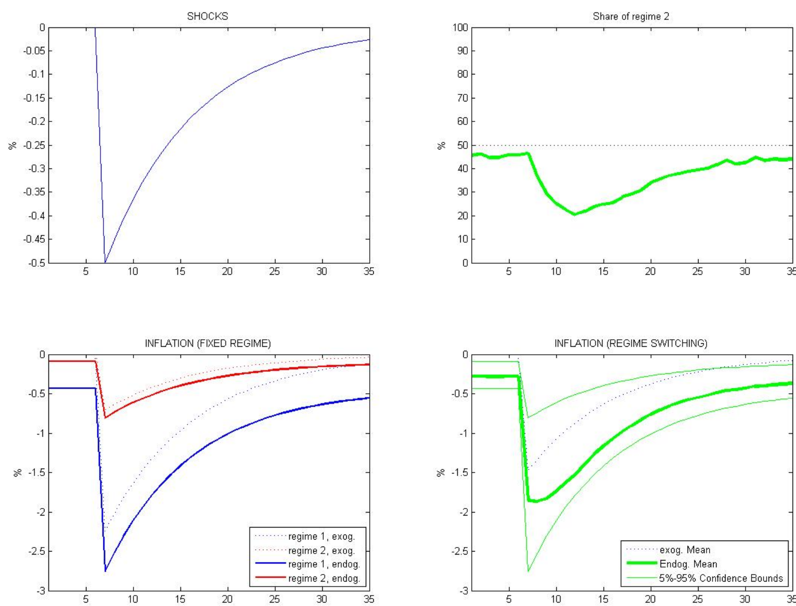

Figure 2: Impulse Response Function to a negative real interest rate shock

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