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A mathematical treatment of bank monitoring incentives[*]

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Abstract

In this paper, we take up the analysis of a principal/agent model with moral hazard introduced in [15], with optimal contracting between a competitive investor and an impatient bank monitoring a pool of long-term loans subject to Markovian contagion. We provide here a comprehensive mathematical formulation of the model and show using martingale arguments in the spirit of Sannikov [17] how the maximization problem with implicit constraints faced by investors can be reduced to a classic stochastic control problem. The approach has the advantage of avoiding the more general techniques based on forward-backward stochastic differential equations described in [6] and leads to a simple recursive system of Hamilton-Jacobi-Bellman equations. We provide a solution to our problem by a verification argument and give an explicit description of both the value function and the optimal contract. Finally, we study the limit case where the bank is no longer impatient.the deal.

Keywords: Default Correlation, Dynamic Moral Hazard, Forward-Backward Stochastic Differential Equations.

JEL Classification: G21, G28, G32.

Résumé

Dans ce papier, nous reprenons l'analyse du modèle principal/agent avec aléa moral décrit dans [15], avec un contrat optimal entre un investisseur compétitif et une banque impatiente, gestionnaire d'un lot de prêts à long terme sujets à contagion Markovienne. Nous formulons le modèle dans sa dimension purement mathématique et montrons par des arguments de martingale à la Sannikov [17] comment le problème de maximisation avec contraintes implicites posé aux investisseurs peut se ramener à un problème classique de contrôle stochastique. L'approche a l'avantage d'éviter les techniques plus générales fondées sur les équations différentielles stochastiques du type « forward-backward » décrites en [6] et de déboucher sur un système récursif simple d'équations d'Hamilton-Jacobi-Bellman. Nous apportons une solution au problème par un argument de vérification et donnons une description explicite tant de la fonction valeur que du contrat optimal. Finalement, nous étudions le cas limite où la banque cesse d'être impatiente.

Mots-clés : Aléa moral dynamique, corrélation des défauts, équations différentielles stochastiques forward-backward.

Codes JEL : G21, G28, G32.

1 Introduction

Following the seminal contributions of DeMarzo and Fishman [10], [11] and Sannikov [17], there has been a renewed interest in the mathematical treatment of continuous-time moral hazard models and their applications. In a typical moral hazard situation, a principal (who takes the initiative of the contract) is imperfectly informed about the action of an agent (who accepts or rejects the contract). The goal is to design a contract that maximizes the utility of the principal while that of the agent is held to a given level.

In its whole generality, the mathematical treatment of the problem can be cast as follows. Agency problems stemming from the agent's hidden action a limit the utility this agent can get from contracting with the principal. The optimal contract c specifies how these limitations should be strenghtened or slackened over time as a result of the agent's ongoing performance. We first have to solve the agent's problem for a given contract

$$V_A(c) := \sup_a \mathbb{E}\left[U_A(c, a)\right],$$

where U_A is the utility function of the agent. If we assume for simplicity that there exists a unique optimal action $a(c)$ for any c, a point on the set of constrained Pareto optima can be found by solving the Principal's stochastic control problem

$$V_P := \sup_c \left\{\mathbb{E}\left[U_P(c, a(c))\right] + \lambda \mathbb{E}\left[U_A(c, a(c))\right]\right\},$$

where U_P is the utility function of the principal and λ is the Lagrange multiplier associated to some reservation utility of the agent.

Because of the almost limitless choices for c, it is generally assumed that the agent does not have complete control over the outcomes but instead continuously affects their distribution by choosing specific actions. This actually means that the agent affects the probability measure $\mathbb{P}^a$ under which the above expectations are taken. This setting, which will be described more rigorously in the following section, corresponds to a weak formulation of the stochastic control problem.

As shown in [6], a general theory can be used to solve these problems, by means of forward-backward stochastic differential equations. We show here how recursive, martingale representation-based techniques proposed by Sannikov [17] can be brought to bear on the issue to yield explicit solutions that are easier to derive. The model we consider, introduced by Pagès in [15], is a contribution to the optimal design of securitization in the presence of banks' impaired incentives to monitor. In contrast with the main thread of the literature, which deals with Brownian motion risk, the focus is on large but infrequent risk, as in Biais et al [3]. An important difference is that credit risk arises in a non-stationary context as the result of imperfectly correlated defaults. Our aim is to provide a coherent mathematical framework for this problem and provide the rigorous foundations for the formal derivations adumbrated in [15].

The rest of the paper is organized as follows. In section 2, we recall the model laid out in [15], describe the contracts and give our main assumptions. In Section 3, we formally derive

a candidate optimal contract by solving the HJB equation associated to the control problem. We then use a standard verification argument to show that the candidate solution is indeed the optimal contract. The paper concludes with a short Section devoted to a simple special case.

2 The model

2.1 Notations and preliminaries

We consider a model with universal risk neutrality in which time is continuous and indexed by $t \in [0, \infty)$. Without loss of generality, the risk-free interest rate is taken to be 0. A bank has a claim to a pool of I unit loans indexed by $i = 1, \ldots, I$ which are ex ante identical. Each loan is a defaultable perpetuity yielding cash flow μ per unit time until it defaults. Once a loan defaults it gives no further payments. The infinite maturity and no recovery assumptions are made for tractability.

Denote by

$$N_t = \sum_{i=1}^{I} 1_{\{\tau^i \leq t\}},$$

the sum of individual loan default indicators, where τ^i denotes the default time of loan i. The current size of the pool is $I - N_t$. Since all loans are a priori identical, they can be reindexed in any order after defaults. The action of the bank consists in deciding at each time t whether it monitors any of the outstanding loans. These actions are summarized by the functions e_t^i defined by

For $1 \leq i \leq I - N_t$, $e_t^i = 1$ if loan i is monitored at time t, and $e_t^i = 0$ otherwise.

Non-monitoring renders a private benefit $B > 0$ per loan and per unit time to the bank. The opportunity cost of monitoring is thus proportional to the number of monitored loans.

The rate at which loan i defaults is controlled by the hazard rate α_t^i specifying its instantaneous probability of default conditional on history up to time t. Individual hazard rates are assumed to depend both on the monitoring choice of the bank and on the size of the pool. Specifically, we choose to model the hazard rate of a non-defaulted loan i at time t as

$$\alpha_t^i = \alpha_{I-N_t} \left(1 + (1 - e_t^i)\varepsilon \right), \tag{2.1}$$

where the parameters $\{\alpha_j\}_{1 \leq j \leq I}$ represent individual "baseline" risk under monitoring when the number of loans is j and ε is the proportional impact of shirking on default risk.

We define the shirking process k by

$$k_t = \sum_{i=1}^{I-N_t} \left(1 - e_t^i \right),$$

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which represents the number of loans that the bank fails to monitor at time t. Then, according to (2.1), aggregate default intensity is given by

$$\lambda_t^k = \alpha_{I-N_t} \left(I - N_t + \varepsilon k_t \right). \tag{2.2}$$

The bank can fund the pool internally at a cost $r \geq 0$. Positive internal funding costs reflect bank's limited access to capital or deposits and may include any regulatory or agency costs associated with this source of financing. The bank can also raise funds from a competitive investor who values income streams at the prevailing riskless interest rate of zero. We assume that both the bank and investors observe the history of defaults and liquidations.

2.2 Description of the contracts

Contracts are offered on a take-it-or-leave-it basis by investors to the bank and agreed upon at time 0. They determine how cash flows are shared and how loans are liquidated, conditionally on past defaults and liquidations. Without loss of generality, they specify that an investor receives cash flows from the pool and makes transfers to the bank. We denote by $D = \{D_t\}_{t \geq 0}$ the càdlàg, positive and increasing process describing cumulative transfers from the investors to the bank, such that

$$\mathbb{E}^{\mathbb{P}} \left[D_\tau \right] < +\infty, \tag{2.3}$$

where τ is the liquidation time of the pool and where we assumed that $D_0 = 0$.

Remark 1. *In certain cases, it can be useful to let D have a jump at time 0. Indeed, for instance in the so called first-best (that is to say when the bank and the investors cooperate), it can be shown that the optimal solution to our problem is to make a lump transfer to the bank at time $0-$ and nothing afterwards. See Remark 3.4 for more details.*

Let then $H_t := 1_{\{t \geq \tau\}}$ be the liquidation indicator of the whole pool. The contract specifies the probability θ_t with which the pool is maintained given default $(dN_t = 1)$, so that at each point in time

$$dH_t = \begin{cases} 0 & \text{with probability } \theta_t, \\ dN_t & \text{with probability } 1 - \theta_t. \end{cases}$$

With our notations, the hazard rates associated with the default and liquidation processes N_t and H_t are λ_t^k and $(1 - \theta_t) \lambda_t^k$, respectively.

The contract also specifies when liquidation occurs. We assume that liquidations can only take the form of the stochastic liquidation of all loans following immediately default. The above properties translate into

$$\mathbb{P} \left(\tau \in \{\tau^1, ..., \tau^I\} \right) = 1, \text{ and } \mathbb{P}(\tau = \tau^i | \mathcal{F}_{\tau^i}, \tau > \tau^{i-1}) = 1 - \theta_{\tau^i}.$$

We summarize the above details of the contracts, which are completely specified by the choice of (D, θ). Each infinitesimal time interval $(t, t + dt)$ unfolds as follows:

- $I - N_t$ loans are performing at time t.

- The bank chooses to leave $k_t \leq I - N_t$ loans unmonitored and monitors the $I - N_t - k_t$ others, enjoying private benefits $k_t B \, dt$.

- The investor receives $(I - N_t) \, \mu \, dt$ from the cash flows generated by the pool and pays $\delta_t dt \geq 0$ as fees to the bank.

- With probability $\lambda_t^k \, dt$ defined by (2.2) there is a default $(dN_t = 1)$.

- Given default the pool is maintained $(dH_t = 0)$ with probability θ_t or liquidated $(dH_t = 1)$ with probability $1 - \theta_t$.

As recalled in the introduction, we assume that the bank's monitoring decision is not observable. This leads to a dynamic moral hazard problem, where the contract (δ, θ) uses observations on defaults to give the bank incentives to monitor. We assume that both the bank and investors can fully commit to such contracts.

2.3 Economic assumptions

In this section we give some Assumptions arising from economic considerations (see [15] for details). They are in force throughout the paper. Let $\overline{\alpha}_j$ denote the harmonic mean of α_1, $\ldots$, α_j, i.e.,

$$\frac{1}{\overline{\alpha}_j} := \frac{1}{j} \sum_{i=1}^{j} \frac{1}{\alpha_i}.$$

Assumption 2.1.

$$\mu \geq \overline{\alpha}_I. \tag{2.4}$$

The condition ensures that monitored loans are profitable viewed as of time 0.

Assumption 2.2. *We have for all $j \leq I$*

$$\frac{r}{\overline{\alpha}_j} \leq \frac{\mu \varepsilon - B}{B} \frac{\varepsilon}{1 + \varepsilon},$$

The condition is related to the efficiency of monitoring and ensures that the benefits for a non-monitoring bank are not so high that shirking is socially preferable.

Assumption 2.3. *Individual default risk is non-decreasing with past default*

$$\alpha_j \leq \alpha_{j-1}, \qquad \text{for all } j \leq I. \tag{2.5}$$

The condition introduces the possibility of correlated defaults through a contagion effect, as individual loans' intensity of default may increase with the arrival of new defaults.

3 Optimal contracting

Before going on, let us now describe the stochastic basis on which we are working. We will always place ourselves on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ on which N is a Poisson process whith intensity λ_t^0 (which is defined by (2.2)) and where $\mathbb{P}$ is the reference probability measure. We denote $(\mathcal{F}_t^N)_{t \geq 0})$ the completed natural filtration of N and by $(\mathcal{G}_t)_{t \geq 0}$ the minimal filtration containing $(\mathcal{F}_t^N)_{t \geq 0})$ and that makes the liquidation time of the pool τ a $\mathcal{G}$-stopping time. We note that this filtration satisfies the usual hypotheses, and therefore we will always consider super or submartingales in their càdlàg version.

3.1 Incentive compatibility and limited liability

As recalled in the introduction, in order to make the problem tractable, we assume that the monitoring choices of the bank affect the distribution of the size of the pool. To formalize this, recall that, by definition, the shirking process k is $\mathcal{G}-$predictable and bounded. Then, by Girsanov Theorem, we can define a probability measure $\mathbb{P}^k$ equivalent to $\mathbb{P}$ such that

$$N_t - \int_0^t \lambda_t^k ds,$$

is a $\mathbb{P}^k$-martingale.

More precisely, we have from Brémaud [4] (Chapter VI, Theorem T3) that on $\mathcal{G}_t$

$$\frac{d\mathbb{P}^k}{d\mathbb{P}} = Z_t^k,$$

where Z^k is the unique solution of the following SDE

$$Z_t^k = 1 + \int_0^t Z_{s^-}^k \left(\frac{\lambda_s^k}{\lambda_s^0} - 1 \right) \left(dN_s - \lambda_s^0 ds \right), \ 0 \leq t \leq T, \ \mathbb{P} - a.s.$$

Then, given a contract (D, θ) and a shirking process k, the bank's expected utility at $t = 0$ is given by

$$u_0^k(D, \theta) := \mathbb{E}^{\mathbb{P}^k} \left[\int_0^\tau e^{-rt} (dD_t + B k_t \, dt) \right], \tag{3.1}$$

while that of the investor is

$$v_0^k(D, \theta) := \mathbb{E}^{\mathbb{P}^k} \left[\int_0^\tau (I - N_t) \, \mu \, dt - dD_t \right]. \tag{3.2}$$

Following Sannikov [17], we give now the definition of an incentive-compatible shirking process.

Definition 1. *A shirking decision k is incentive-compatible with respect to the contract (D, θ) if it maximizes (3.1).*

Then, the problem faced by the investors is to design a contract (D, θ) and an incentive-compatible advice on k that maximize their expected discounted payoff, subject to a given reservation utility for the bank

$$v_I(u) := \sup_{(D,\theta,k)} \mathbb{E}^{\mathbb{P}^k}\left[\int_0^\tau (I - N_t)\,\mu dt - dD_t\right] \tag{3.3}$$

$$\text{subject to} \quad \mathbb{E}^{\mathbb{P}^k}\left[\int_0^\tau e^{-rt}(dD_t + Bk_t\,dt)\right] \geq u$$

$$k \text{ incentive-compatible with respect to } (D, \theta).$$

This allows us to define a first set of admissible contracts for a given monitoring advice k

$$\mathcal{A}^k(x) := \{(D, \theta),\ \theta \text{ is a predictable process with values in } [0, 1],$$

$$D \text{ is a positive càdlàg non-decreasing process which satisfies (2.3)},$$

$$k \text{ is incentive-compatible with respect to } (D, \theta) \text{ and } u_0^k(D, \theta) \geq x\}. \tag{3.4}$$

Notice that we will put more restrictions on this set at the end of the section.

Using martingale arguments, we now elicit an equivalent condition for the incentive compatibility of k. Consider the bank's expected lifetime utility, conditional on $\mathcal{G}_t$

$$U_t^k(D, \theta) := \mathbb{E}^{\mathbb{P}^k}\left[\int_0^\tau e^{-rs}\,(dD_s + Bk_s ds)\ \Big|\ \mathcal{G}_t\right] \tag{3.5}$$

$$= \int_0^{t\wedge\tau} e^{-rs}\,(dD_s + Bk_s ds) + e^{-rt}u_t^k(D, \theta),$$

where u_t^k is the dynamic version of the bank's continuation utility defined as

$$u_t^k(D, \theta) := \mathbb{1}_{\{t<\tau\}}\mathbb{E}^{\mathbb{P}^k}\left[\int_t^\tau e^{-r(s-t)}\,(dD_s + Bk_s ds)\ \Big|\ \mathcal{G}_t\right]. \tag{3.6}$$

Since we are working with the completed natural filtration of a Poisson process, and since U_t^k is a $\mathcal{G}_t$-martingale under $\mathbb{P}^k$ and in L^1 because of the integrability assumptions we made, the martingale representation theorem for point processes (see [4], Chapter III, Theorems $T9$ and $T17$, and Chapter VI, Theorems $T2$ and $T3$) implies that there are predictable processes h^1 and h^2 such that the bank's continuation utility u^k satisfies the following "promise-keeping" equation until liquidation occurs

$$du_t^k + (dD_t + Bk_t dt) = ru_t^k\,dt - h_t^1\left(dN_t - \lambda_t^k\,dt\right) - h_t^2\left(dH_t - (1 - \theta_t)\lambda_t^k\,dt\right), \tag{3.7}$$

where the dependence of h^1 and h^2 on k has been suppressed for notational convenience.

The introduction of the processes h^1 and h^2 provides a practical way of characterizing contracts for which a given k is incentive-compatible, as shown in the following proposition, inspired by Sannikov [17].

Proposition 1. *Given a contract (D, θ) and a shirking process k, the latter is incentive-compatible if and only if for all $t \in [0, \tau]$ and for all $i = 1...I$, the following holds almost-surely,*

$$\left(\frac{B}{\varepsilon \alpha_{I-N_t}} - h_t^1 - (1 - \theta_t) h_t^2 \right) (k_t - i) \geq 0. \tag{3.8}$$

Proof. Consider an arbitrary strategy $\widehat{k}$ specifying the number of unmonitored loans at any point in time until liquidation. Let u_t^k denote the continuation utility in (3.6) resulting from the decision to forgo monitoring k loans at all times.

$$\widehat{U}_t = \int_0^{t \wedge \tau} e^{-rs} \left(dD_s + B \widehat{k}_s ds \right) + e^{-rt} u_t^k \tag{3.9}$$

the lifetime utility of the bank viewed as of time t if it follows the strategy $\widehat{k}$ before time t, and plans to switch to k afterwards.

We have for all $t \in [0, \tau]$

$$
\begin{aligned}
d\widehat{U}_t &= e^{-rt} \left(dD_t + B \widehat{k}_t dt \right) + e^{-rt} \left(du_t^k - r u_t^k \, dt \right) \\
&= e^{-rt} B (\widehat{k}_t - k_t) \, dt - e^{-rt} \left(h_t^1 (dN_t - \lambda_t^k \, dt) + h_t^2 (dH_t - (1 - \theta_t) \lambda_t^k \, dt) \right) \\
&= e^{-rt} \left(B - \alpha_{I-N_t} \varepsilon (h_t^1 + (1 - \theta_t) h_t^2) \right) (\widehat{k}_t - k_t) dt \\
&\quad - e^{-rt} \left(h_t^1 (dN_t - \lambda_t^{\widehat{k}} \, dt) + h_t^2 (dH_t - (1 - \theta_t) \lambda_t^{\widehat{k}} \, dt) \right),
\end{aligned}
$$

where we have used the promise-keeping equation (3.7) for u^k. Therefore, the first term on the right-hand side

$$e^{-rt} \left(B - \alpha_{I-N_t} \varepsilon (h_t^1 + (1 - \theta_t) h_t^2) \right) (\widehat{k}_t - k_t),$$

is the drift of $\widehat{U}$ under $\mathbb{P}^{\widehat{k}}$. Note also that, by definition, h^1 and h^2 are integrable and therefore the martingale part of $\widehat{U}$ is a true $\mathbb{P}^{\widehat{k}}$-martingale.

(i) Now assume that (3.8) does not hold on a set of positive measure, and choose $\widehat{k}$ such that it maximizes the quantity

$$\left(B - \alpha_{I-N_t} \varepsilon (h_t^1 + (1 - \theta_t) h_t^2) \right) \widehat{k}_t,$$

for all t.

Then, the drift of $\widehat{U}$ under $\mathbb{P}^{\widehat{k}}$ is non-negative and strictly positive on a set of positive measure. Therefore $\widehat{U}$ is a $\mathbb{P}^{\widehat{k}}$-submartingale. This implies the existence of a time $t^* > 0$ such that

$$\mathbb{E}^{\mathbb{P}^{\widehat{k}}}[\widehat{U}_{t^*}] > \widehat{U}_0 = u_0^k.$$

Therefore, if the agent follows this strategy $\widehat{k}$ until the time t^* and then switches to the strategy k, his utility is strictly greater than the utility obtained from following the strategy k all the time. This contradicts the fact that the strategy k is incentive-compatible.

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(ii) With the same notations as above, assume that (3.8) holds for the strategy k. Then this means that $\widehat{U}$ is a $\mathbb{P}^{\widehat{k}}$-supermartingale, regardless of the choice of strategy $\widehat{k}$. Moreover, since $\widehat{U}$ is positive (because D is non-decreasing), it has a last element (see Problem 3.16 in [12] for instance). Then, we have by the optional sampling Theorem

$$u_0^k = \widehat{U}_0 \geq \mathbb{E}^{\mathbb{P}^{\widehat{k}}}\left[\widehat{U}_\tau\right] = u_0^{\widehat{k}},$$

where we used (3.9) and the fact that $u_\tau^k = 0$ for the last inequality.

This means that the strategy k maximizes the expected utility of the agent and is therefore incentive-compatible. □

Under the assumption that monitoring is efficient, we now focus on contracts that actually deter the bank from shirking, i.e., contracts with respect to which $k = 0$ is incentive-compatible. In that particular case, the above Proposition can be simplified as follows.

Corollary 2. *Given a contract* (D, θ), $k = 0$ *is incentive-compatible if and only if*

$$h_t^1 + (1 - \theta_t)h_t^2 \geq \frac{B}{\varepsilon \alpha_{I-N_t}}, \ t \in [0, \tau], \ \mathbb{P} - a.s. \tag{3.10}$$

Remark 2. *Corollary 2 states that, given that the pool has i loans outstanding, in order to induce the bank to monitor all loans, the continuation payoff must drop in expectation by at least the quantity*

$$b_i := \frac{B}{\varepsilon \alpha_i},$$

following default.

In order to specify further our admissible strategies, we have to put some restrictions on h^1 and h^2. First, we assume that the bank has limited liability. This means that the bank's continuation utility is bounded from below by b_{I-N_t} up to liquidation, since otherwise the incentive-compatible (3.10) would be violated upon default.

The limited liability constraint must also holds after a default if the pool is maintained in operation $(dH_t = 0)$, when the drop in utility is h^1. This implies that the following condition holds

$$\text{For all } 1 \leq i \leq I, \ u_{t-}^0 - h_t^1 \geq b_{i-1}, \ \text{on } \{N_t = I - i\}. \tag{3.11}$$

For the second condition, we assume that the bank forfeits any rights to cash flows once the pool is liquidated. The constraint $u_\tau^0 = 0$ implies in turn that at all times

$$u_{t-}^0 = h_t^1 + h_t^2, \tag{3.12}$$

since the drop in utility is $h^1 + h^2$ in that case.

The introduction of the processes h^1 and h^2 allows us to greatly simplify the set of admissible contracts by formulating the incentive compatibility requirement in terms of explicit conditions. Our set of admissible strategies is therefore

$\widetilde{\mathcal{A}}^0(x) := \{(D, \theta, h^1, h^2),\ \theta$ is a predictable process with values in $[0, 1]$,

$\qquad D$ is a positive càdlàg non-decreasing process which satisfies (2.3),

$\qquad h^1$ and h^2 are predictable processes, integrable, and satisfy $u^0_{t-} - h^1_t \geq b_{I-N_t-1}$,

$\qquad u^0_{t-} = h^1_t + h^2_t,\ x \leq u^0_0(D, \theta).\}.$ \hfill (3.13)

3.2 Reduction to a stochastic control problem and HJB equation

Under condition (3.10), $k = 0$ is incentive-compatible. That being taken care of, solving for the optimal contract involves maximizing an investor's expected utility and is therefore a classical stochastic control problem. Let $v_j(u)$ denote the investor's value function, i.e., the maximum expected utility an investor can achieve given a pool of size j and a reservation utility for the bank u. Assume for now that the processes D are absolutely continuous with respect to the Lebesgue measure (we will verify later that the property is satisfied at the optimum), that is to say

$$D_t = \int_0^t \delta_s ds.$$

Then, we expect the investor's value function to solve the following system of Hamilton-Jacobi-Bellman equations with initial condition $v_0(u) = 0$

$$\sup_{(\delta, \theta, h^1, h^2) \in \mathcal{C}^j} \left\{ \left(ru + \lambda_j \left(h^1 + (1 - \theta)h^2 \right) - \delta \right) v'_j(u) + j\mu - \delta \right.$$
$$\left. -\theta \lambda_j \left(v_j(u) - v_{j-1}(u - h^1) \right) - (1 - \theta)\lambda_j v_j(u) \right\} = 0,\ u \geq b_j, \hfill (3.14)$$

where the $\mathcal{C}^j$ are our admissible strategies sets defined by

$$\mathcal{C}^j := \left\{ (\delta, \theta, h^1, h^2),\ \delta \geq 0,\ \theta \in [0, 1],\ h^1 + (1 - \theta)h^2 \geq b_j,\ u - h^1 \geq b_{j-1},\ u = h^1 + h^2 \right\}.$$

Remark 3.1. *We will see in the next section that our control problem is singular. Therefore the above HJB equation (3.14) is not exactly the correct one, and we will consider instead a variational inequality.*

Given the constraints in the definition of $\mathcal{C}^j$, we reparametrize the problem in terms of the variable $z := \theta(u - h^1)$. This leads to the simpler system of HJB equations

$$\sup_{(\delta,\theta,z)\in\widetilde{\mathcal{C}}^j}\left\{\left(ru+\lambda_j\left(u-z\right)-\delta\right)v'_j(u)+j\mu-\delta-\lambda_j\left(v_j(u)-\theta v_{j-1}\left(\frac{z}{\theta}\right)\right)\right\}=0,\ u\geq b_j,$$

(3.15)

where the constraints become

$$\widetilde{\mathcal{C}}^j:=\left\{(\delta,\theta,z),\ \delta\geq0,\ \theta\in\left[0,1\wedge\frac{u-b_j}{b_{j-1}}\right],\ \text{and}\ z\in[b_{j-1}\theta,u-b_j]\right\}.$$

Our strategy now is to guess a candidate optimal contract by solving the above system of HJB equations, and to prove that the conjectured contract is indeed optimal by means of a verification argument. However, since $j=1$ is a degenerate special case, it is convenient to treat monitoring with a single loan first before turning to the general case.

3.3   Single loan: Constant utility

We provide below a solution of the HJB equation which is compatible with our problem, in the sense that the initial conditions for v_1 are obtained from our formulation of the Principal/Agent problem.

Since there is only one loan, when it defaults the pool is automatically liquidated, which means that θ is always equal to 1. Since $v_0=0$ and $b_0=0$, optimizing first with respect to δ yields the following variational inequality for $u>b_1$

$$\min\left\{-\sup_{b_1\leq h^1\leq u}\left\{\left(ru+\lambda_1 h^1\right)v'_1(u)+\mu-\lambda_1 v_1(u)\right\},v'_1(u)+1\right\}=0.$$

(3.16)

Moreover, it appears that $\delta=0$ as long as $v'_1(u)+1>0$. Starting from this, finding the solution is an easy but lengthy exercise so we postpone the corresponding discussion to the Appendix. It nonetheless leads to the following Proposition

Proposition 3.1. *The function v_1 defined by*

$$v_1(u):=b_1-u+\frac{\mu-b_1(r+\lambda_1)}{\lambda_1},\ u>b_1,$$

is a solution of (3.15).

Moreover if we extend linearly this function by continuity on $[0,b_1]$, v_1 is concave.

Remark 3.2. *In the case $j=1$ the utility of the bank is always b_1 and the bank receives constant payments $\delta_t=rb_1+\lambda_1 b_1$ until the loan defaults. We refer to Section 3.6 for the proof that the contract described above is indeed the optimal one when there is only one loan in the pool. We also refer the reader to the next section to understand the utility of extending the function on $[0,b_1]$.*

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3.4 Formal derivation of a candidate optimal contract

In this section, we show how to formally obtain a recursive system of ODEs which should be satisfied by our value function.

Step (i) Optimizing first with respect to δ yields the following variational inequality for $u > b_j$

$$\min\left\{-\sup_{(\theta,z)\in\widetilde{\mathcal{B}}^j}\left\{(ru + \lambda_j(u-z))v_j'(u) + j\mu - \lambda_j\left(v_j(u) - \theta v_{j-1}\left(\frac{z}{\theta}\right)\right)\right\}, v_j'(u) + 1\right\} = 0.$$

(3.17)

where

$$\widetilde{\mathcal{B}}^j := \left\{(\theta, z),\ \theta \in \left[0, 1 \wedge \frac{u-b_j}{b_{j-1}}\right],\ \text{and } z \in [b_{j-1}\theta, u - b_j]\right\}.$$

We continue our guess of the value function assuming that all the functions v_j are concave (a property which needs to be verified by our candidate). Then the first derivative of v_j is decreasing. Let us also assume that there exists a level $\gamma_j > b_j$ (a free boundary) such that

$$v_j'(\gamma_j) = -1,\ v_j'(u) > -1,\ \text{for } u < \gamma_j,$$

Then as long as $u < \gamma_j$, v_j satisfies the first equation in (3.17). Therefore, equation (3.17) tells us that the bank receives cash from the investors only when its utility attains the level γ_j (since $\delta = 0$ is optimal before that). We also assume (and we will verify) that our candidate satisfy

$$-\sup_{(\theta,z)\in\widetilde{\mathcal{B}}^j}\left\{(ru + \lambda_j(u-z))v_j'(u) + j\mu - \lambda_j\left(v_j(u) - \theta v_{j-1}\left(\frac{z}{\theta}\right)\right)\right\} \geq 0,\ u \geq \gamma_j.$$

This means that v_j becomes linear above γ_j, and that the variational inequality (3.17) takes the simpler form

$$-\sup_{(\theta,z)\in\widetilde{\mathcal{B}}^j}\left\{(ru + \lambda_j(u-z))v_j'(u) + j\mu - \lambda_j\left(v_j(u) - \theta v_{j-1}\left(\frac{z}{\theta}\right)\right)\right\} = 0,\ b_j < u \leq \gamma_j$$

$$v_j'(u) + 1 = 0,\ u > \gamma_j.$$

Now in order to know which level γ_j should be chosen, it is natural to require our solution to be maximal in the sense that for each $u > b_j$

$$\gamma_j \longrightarrow v_j(u),$$

is maximal at the chosen value of γ_j. Of course, it is not clear at all whether such a value exists. Nonetheless, we will prove that this heuristic approach can be proven rigorously, and that our maximality assumption has a clear economic meaning.

Step (ii) We next turn to the liquidation decision, one finds as first-order condition with respect to θ

$$v_{j-1}\left(\frac{z}{\theta}\right) - \frac{z}{\theta}v'_{j-1}\left(\frac{z}{\theta}\right) \geq 0. \tag{3.18}$$

Once again, if v_{j-1} is concave, the above inequality (3.18) is always verified. This means that the function

$$\theta \longrightarrow \theta v_{j-1}\left(\frac{z}{\theta}\right),$$

is non-decreasing, which implies that the optimal θ corresponds to its upper bound.

There are then two cases

(i) $u \in [b_j, b_j + b_{j-1})$ and $\theta = \frac{u-b_j}{b_{j-1}}$.

(ii) $u \in \left[b_j + b_{j-1}, \gamma_j\right]$ and $\theta = 1$.

Step (iii) Finally consider the decision regarding z. First, if $u \in [b_j, b_j + b_{j-1})$, then z has to be equal to $u - b_j$. Then, in the probation interval $\theta = 1$ and z is constrained in the range $[b_{j-1}, u - b_j]$. We continue our guess of a candidate solution assuming that

$$v'_{j-1}(u - b_j) - v'_j(u) \geq 0, \tag{3.19}$$

a condition which needs to be verified by the resulting candidate.

Then, since v_{j-1} is supposed to be concave, we have for all $z \in [b_{j-1}, u - b_j]$

$$v'_{j-1}(z) - v'_j(u) \geq 0.$$

From this, we obtain that the function $z \longrightarrow -zv'_j + v_{j-1}(z)$ is non-decreasing, which in turn implies that the supremum over z is also attained at $u - b_j$ in the probation interval.

Summarizing all the above formal calculations, we end up with the following system of ODEs, which should lead us to a solution of the HJB equation on the interval $[b_j, \gamma_j]$

$$(ru + \lambda_j b_j)\, v'_j(u) + j\mu - \lambda_j \left(v_j(u) - v_{j-1}(u - b_j)\right) = 0, \qquad u \in \left(b_j + b_{j-1}, \gamma_j\right]$$

$$(ru + \lambda_j b_j)\, v'_j(u) + j\mu - \lambda_j \left(v_j(u) - \frac{u - b_j}{b_{j-1}}v_{j-1}(b_{j-1})\right) = 0, \qquad u \in (b_j, b_j + b_{j-1}].$$

We next extend the value function v_j to the interval $[0, b_j]$ by setting

$$v_j(u) := \frac{u}{b_j}v_j(b_j), \qquad u \in [0, b_j], \tag{3.20}$$

and to the interval $(\gamma_j, +\infty)$ by

$$v_j(u) := v_j(\gamma_j) - u + \gamma_j.$$

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Then the above system of ODEs simplifies to

$$(ru + \lambda_j b_j) \, v'_j(u) + j\mu - \lambda_j \left(v_j(u) - v_{j-1}(u - b_j) \right) = 0, \qquad u \in \left(b_j, \gamma_j \right]$$
$$v'_j(u) = -1, \; u \geq \gamma_j. \tag{3.21}$$

Recall that we need to verify that the solution obtained from (3.21) satisfies all the properties assumed in the derivation of our candidate.

3.5 Solving the HJB equation

We now provide conditions under which the heuristic derivation of the previous section indeed corresponds to a solution of the original system of HJB equations (3.15). Since we already solved the problem for $j = 1$, we assume here that $j \geq 2$.

Let us define

$$\overline{v}_j := v_j(b_j),$$

and for $x > 0$ and $0 < \beta \leq 1$ the functions

$$\phi_\beta(x) := \left(\frac{1 + x}{1 + (1 + \beta)x} \right)^{\frac{1}{x} - 1}, \quad \psi_\beta(x) := \frac{\phi_\beta(x) - x}{(1 - x)\phi_\beta(x)}.$$

Remark 3.3. *Then, it is easy to prove that the functions ψ_β can be extended to continuous functions on $\mathbb{R}_+$ which decrease from 1 to $\frac{1}{2}$ and that for all $x \geq 0$*

$$\psi_1(x) = \inf_{0 < \beta \leq 1} \psi_\beta(x).$$

We have the following results.

Proposition 3.2. *Assume that*

$$\frac{r}{\lambda_j} - 1 \leq \frac{\overline{v}_{j-1}}{b_{j-1}}. \tag{3.22}$$

(i) *The ordinary differential equations (3.21), along with (3.20), have unique maximal solutions v_j for $j \geq 2$. The functions v_j are globally concave, differentiable everywhere except at b_j and twice differentiable everywhere except at b_j and $b_j + b_{j-1}$. The endogenous thresholds $\gamma_j \geq b_j + b_{j-1}$ are uniquely determined by*

$$\frac{r}{\lambda_j} - 1 \in \partial v_{j-1}(\gamma_j - b_j), \tag{3.23}$$

where $\partial v_j(u)$ is the subdifferential of v_j at u and verify

$$\gamma_j \leq b_j + \gamma_{j-1}. \tag{3.24}$$

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(ii) *The λ_j can be chosen recursively so that*

$$\left(v'_{j-1}(b^+_{j-1})\right)^+ \frac{b_{j-1}}{\bar{v}_{j-1}} \leq \psi_1\left(\frac{r}{\lambda_j}\right), \tag{3.25}$$

In that case, the functions v_j also verify

$$v'_j(u) - v'_{j-1}(u - b_j) \leq 0, \text{ for all } u \geq b_j. \tag{3.26}$$

The proof is rather tedious and is relegated to the Appendix.

Now since the functions v_j constructed in Proposition 3.2 are globally concave, have a derivative which is greater than -1 for $u < \gamma_j$ and equal to -1 for $u \geq \gamma_j$ and satisfy (3.26), we can apply the heuristic arguments of Section 3.4 to obtain the following corollary.

Corollary 3.1. *Under the assumptions of Proposition 3.2, the functions v_j constructed in the same Proposition solve the HJB equation (3.14).*

Proof. The only remaining property to prove is that for $u \geq \gamma_j$, we have

$$-(ru + \lambda_j b_j) v'_j(u) - j\mu + \lambda_j\left(v_j(u) - v_{j-1}(u - b_j)\right) \geq 0.$$

We compute

$$
\begin{aligned}
&-(ru + \lambda_j b_j) v'_j(u) - j\mu + \lambda_j\left(v_j(u) - v_{j-1}(u - b_j)\right) \\
&= ru + \lambda_j b_j - j\mu + \lambda_j\left(v_j(\gamma_j) - u + \gamma_j - v_{j-1}(u - b_j)\right) \\
&= r(u - \gamma_j) + \lambda_j\left(v_{j-1}(\gamma_j - b_j) + \gamma_j - b_j - v_{j-1}(u - b_j) - u + b_j\right) \\
&\geq r(u - \gamma_j) - \lambda_j(u - \gamma_j)\left(1 + \sup_{\gamma_j - b_j \leq x \leq u - b_j} v'_{j-1}(x)\right) \\
&\geq r(u - \gamma_j) - \lambda_j(u - \gamma_j)\frac{r}{\lambda_j} \\
&= 0,
\end{aligned}
$$

where we used the fact that v_{j-1} is concave, that $u \to v_{j-1}(u) + u$ is increasing and that $v'_{j-1}(\gamma_j - b_j) \leq \frac{r}{\lambda_j} - 1$.

In particular, this shows that

$$
\begin{aligned}
-\sup_{(\theta, h^1, h^2) \in \mathcal{B}^j} &\left\{\left(ru + \lambda_j\left(h^1 + (1 - \theta)h^2\right)\right) v'_j(u) + j\mu \right. \\
&\left. -\lambda_j\left(v_j(u) - \theta v_{j-1}(u - h^1)\right)\right\} \geq 0, \ u \geq \gamma_j. \tag{3.27}
\end{aligned}
$$

$\square$

Let us finally describe the contract (D, θ), first obtained in [15], which can be deduced from the above results. Starting from a reservation utility $x \leq \gamma_I$ for the bank, the following contract unfolds.

Contract 3.1. (i) *Given size j, the pool remains in operation (i.e. there is no liquidation) with one less unit at any time there is a default in the range $[b_j + b_{j-1}, \gamma_j]$.*

(ii) *The flow of fees paid to the bank given j is $\delta_t^j = \lambda_j b_j + r\gamma_j$ as long as $u_t = \gamma_j$ and no default occurs, where δ^j is the density of D with respect to the Lebesgue measure. Otherwise $\delta_t = 0$.*

(iii) *Liquidation of the whole pool occurs with probability $\theta_t^j = (u_t - b_j)/b_{j-1}$ in the range $[b_j, b_j + b_{j-1})$.*

To summarize, we have for j given and with the original notations of (3.14)

$$\delta^j(u) := 1_{u=\gamma_j}(\lambda_j b_j + r\gamma_j)$$

$$\theta^j(u) := 1_{b_j \leq u < b_j + b_{j-1}}\frac{u - b_j}{b_{j-1}} + 1_{b_j + b_{j-1} \leq u \leq \gamma_j}$$

$$h^{1,j}(u) := (u - b_{j-1})1_{b_j \leq u < b_j + b_{j-1}} + b_j 1_{b_j + b_{j-1} \leq u \leq \gamma_j}$$

$$h^{2,j}(u) := u - h^{1,j}(u). \tag{3.28}$$

Remark 3.4. *If the reservation utility for the bank x is greater than γ_I then the contract should specify in addition that a transfer is immediately made to the bank so that its utility returns to the level γ_I. This means that instead of considering transfers $(D_t)_{t\geq 0}$ which are only absolutely continuous with respect to the Lebesgue measure, we have to add a Dirac mass at 0. Our proofs can then be easily adjusted to that case, therefore we will not treat it. Moreover, notice that the contract 3.1 is clearly in $\widetilde{\mathcal{A}}^0(x)$.*

3.6 The verification theorem

In this subsection, we prove our main result.

Theorem 3.1. *Let $u_0 \leq \gamma_I$ be the reservation utility for the bank. Then, the optimal contract in $\widetilde{\mathcal{A}}^0(x)$ for the problem (3.3) is the contract 3.1.*

We decompose the proof in two parts. First, we show that the bank can obtain a level of utility u_0 and the investors $v_I(u_0)$, for any $u_0 \geq b_I$, with this contract. The second part, reported in Proposition 3.3, shows that for any contract (D, θ) which makes the shirking decision $k = 0$ incentive-compatible, the utility the investors can obtain is bounded from above by $v_I(u_0)$, where u_0 is the utility obtained by the bank.

Proposition 3. *Let the assumptions of Proposition 3.2 hold true. For any starting condition $u_0 > b_I$, we define the process u_t as the solution of the following SDE for $j = 0, \ldots, I - 1$*

$$du_t = (ru_t - \delta^{I-N_t}(u_t))\, dt - h^{1,I-N_t}(u_t)(dN_t - \lambda_{I-N_t}\, dt)$$

$$- h^{2,I-N_t}(u_t)(dH_t - \lambda_{I-N_t}(1 - \theta^{I-N_t}(u_t))\, dt),\ t < \tau. \tag{3.29}$$

Then, the contract defined by $\left(\delta^{I-N_t}(u_t), \theta^{I-N_t}(u_t)\right)$ is incentive-compatible, has value u_0 for the bank and value $v_I(u_0)$ for the investors.

Proof. First, the drift and volatility in the SDE (3.29) are clearly Lipschitz. This guarantees the existence and uniqueness of the solution for all t. Moreover, it is also clear from the definitions of δ^{I-N_t}, θ^{I-N_t}, $h^{1,I-N_t}$ and $h^{2,I-N_t}$ that

$$ru_t - \delta^{I-N_t} + \lambda_{I-N_t}\left(h^{1,I-j}(u_t) + (1 - \theta^{I-N_t}(u_t))h^{2,I-N_t}(u_t)\right) \geq 0.$$

Hence u_t remains below γ_{I-N_t}. Moreover, when N jumps, we have at the time of the jump

$$
\begin{aligned}
u_t &= u_{t-} - h_t^{1,I-N_{t-}} \\
&= b_{I-N_{t-}} \mathbf{1}_{b_{I-N_{t-}} \leq u_{t-} < b_{I-N_t} + b_{I-N_{t-}}} + (u_{t-} - b_{N_{t-}})\mathbf{1}_{b_{I-N_t} + b_{I-N_{t-}} \leq u_{t-} \leq \gamma_{I-N_{t-}}} \\
&\geq b_{I-N_t}.
\end{aligned}
$$

Therefore, we always have $u_t \geq b_{I-N_t}$ for $t < \tau$. Hence, the process u is bounded.

Moreover, it is clear by construction that this contract makes the shirking decision $k = 0$ incentive-compatible. Indeed, we have after some calculations for all j

$$h^{1,I-N_t}(u_t) + (1 - \theta^{I-N_t}(u_t))h^{2,I-N_t}(u_t) = b_{I-N_t}, \ t < \tau,$$

which is exactly (3.10).

Then, using the equation (3.7) for the continuation utility of the bank obtained with the contract $(\delta^{I-N_t}(u_t), \theta^{I-N_t}(u_t))$, we obtain

$$
\begin{aligned}
d\left(e^{-rt}(u_t^0 - u_t)\right) &= e^{-rt}\left((h_t^1 - h^{1,I-N_t}(u_t))(dN_t - \lambda_{I-N_t}dt)\right) \\
&\quad + e^{-rt}\left((h_t^2 - h^{2,I-N_t}(u_t))(dH_t - \lambda_{I-N_t}(1 - \theta^{I-N_t}(u_t))dt)\right).
\end{aligned}
$$

Since $h^{1,N_t}(u_t)$ and $h^{2,I-N_t}(u_t)$ are bounded because u_t is bounded and since h_t^1 and h_t^2 are in the space $L^1(\mathbb{P})$ by construction, we can take the conditionnal expectation above to obtain

$$\mathbb{E}_t\left[u_{t+s}^0 - u_{t+s}\right] = e^{rs}(u_t^0 - u_t).$$

u^0 remains bounded, because the δ^j are bounded for all j (recall (3.6)) and u is bounded, thus the left-hand side above must remain bounded. Since $r > 0$, letting s go to $+\infty$ implies that $u_t = u_t^0$, $\mathbb{P} - a.s.$ and in particular that the bank overall utility is

$$u_0^0 = u_0.$$

Let us now turn our attention to the investors. Define

$$G_t := \int_0^t ((I - N_s)\mu - \delta(u_s))ds + v_{I-N_t}(u_t), \tag{3.30}$$

where the v_j are those defined in Proposition 3.2.

Let us place ourselves on the interval $[\tau_j \wedge \tau, \tau_{j+1} \wedge \tau)$. We have shown before that u_t remains above b_{I-j}. But we know by construction that v_{I-j} is continuous on $[b_{I-j}, +\infty)$ and has a derivative which can be continuously extended on $[b_{I-j}, +\infty)$. Hence we can apply the change of variable formula for locally bounded processes (see [8], Chapter VI, Section 92) to obtain for all $t \geq 0$

$$G_t = v_I(u_0) + \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} (I-j)\mu - \delta^{I-j}(u_s) + v'_{I-j}(u_s) \left(r u_s - \delta^{I-j}(u_s)\right) ds$$

$$+ \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} \lambda_{I-j} v'_{I-j}(u_s) \left(h^{1,I-j}(u_s) + (1 - \theta^{I-j}(u_s))h^{2,I-j}(u_s)\right) ds$$

$$+ \sum_{j=0}^{I-1} \sum_{\tau_j \wedge t \leq s \leq \tau_{j+1} \wedge t} v_{I-j}(u_s) - v_{I-j}(u_{s-}). \tag{3.31}$$

Let us decompose the jumps of v_j. We have

$$v_j(u_s) - v_j(u_{s-}) = \Delta N_s \left((1 - \Delta H_s)v_{j-1}\left(u_{s-} - h^{1,j}(u_{s-})\right) - v_j(u_{s-})\right)$$

$$= \Delta N_s \left(v_{j-1}\left(u_{s-} - h^{1,j}(u_{s-})\right) - v_j(u_{s-})\right) - \Delta H_s v_{j-1}\left(u_{s-} - h^{1,j}(u_{s-})\right),$$

which implies that

$$\sum_{\tau_j \wedge t \leq s \leq \tau_{j+1} \wedge t} v_{I-j}(u_s) - v_{I-j}(u_{s-}) = \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} \left(v_{I-j-1}\left(u_{s-} - h^{1,I-j}(u_{s-})\right) - v_{I-j}(u_{s-})\right) dN_s$$

$$- \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} v_{I-j-1}\left(u_{s-} - h^{1,I-j}(u_{s-})\right) dH_s.$$

From this, we obtain

$$G_t = v_I(u_0) + \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} (I-j)\mu - \delta^{I-j}(u_s) + v'_{I-j}(u_s)\left(ru_s - \delta^{I-j}(u_s)\right) ds$$

$$+ \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} \lambda_{I-j} v'_{I-j}(u_s)\left(h^{1,I-j}(u_s) + (1-\theta^{I-j}(u_s))h^{2,I-j}(u_s)\right) ds$$

$$+ \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} \lambda_{I-j}\left(v_{I-j-1}\left(u_s - h^{1,I-j}(u_s)\right) - v_{I-j}(u_s)\right) ds$$

$$- \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} \lambda_{I-j}(1-\theta^{I-j})v_{I-j-1}\left(u_s - h^{1,I-j}(u_s)\right) ds$$

$$+ \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} \left(v_{I-j-1}\left(u_{s^-} - h^{1,I-j}(u_{s^-})\right) - v_{I-j}(u_{s^-})\right)(dN_s - \lambda_{I-j}ds)$$

$$- \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} v_{I-j-1}\left(u_{s^-} - h^{1,I-j}(u_{s^-})\right)\left(dH_s - \lambda_{I-j}(1-\theta^{I-j}(u_{s^-}))ds\right).$$

Using the fact that the v_j solve the HJB equation 3.21, we deduce that

$$G_t = v_I(u_0) + \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} \left(v_{I-j-1}\left(u_{s^-} - h^{1,I-j}(u_{s^-})\right) - v_{I-j}(u_{s^-})\right)(dN_s - \lambda_{I-j}ds)$$

$$- \sum_{j=0}^{I-1} \int_{\tau_j \wedge t}^{\tau_{j+1} \wedge t} v_{I-j-1}\left(u_{s^-} - h^{1,I-j}(u_{s^-})\right)\left(dH_s - \lambda_{I-j}(1-\theta^{I-j}(u_{s^-}))ds\right). \quad (3.32)$$

Hence, G is a bounded martingale until time τ (since δ is bounded by definition and u_t and thus the $v_j(u_t)$ are also bounded) and we have, since $u_\tau = 0$

$$\mathbb{E}\left[\int_0^\tau ((I-N_t)\mu - \delta_t)\, dt\right] = \mathbb{E}[G_\tau] = G_0 = v_I(u_0),$$

which is the desired result. $\qquad \square$

We now show that $v_I(u_0)$ is an upper bound for the utility the investor can obtain from any contract which makes the shirking decision $k = 0$ incentive-compatible.

Proposition 3.3. *For any contract $(D, \theta) \in \widetilde{\mathcal{A}}^0(u_0)$, the utility the investors can obtain is bounded from above by $v_I(u_0)$, where u_0 is the utility obtained by the bank.*

Proof. We define as in the previous proof the quantity G_t for an arbitrary contract (δ, θ). By applying the change of variable formula and arguing exactly as before we can obtain that the drift of G is actually negative, using again (3.14). Indeed, we know that for any $(D, \theta, h^1, h^2) \in \widetilde{\mathcal{A}}^0(u_0)$, we have from Corollary 3.1 and its proof that for all j

$$\left(ru_t + \lambda_j\left(h_t^1 + (1-\theta_t)h_t^2\right)\right)v'_j(u_t) + j\mu - \lambda_j\left(v_j(u_t) - \theta_t v_{j-1}(u_t - h_t^1)\right) \le 0,$$

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and we know that
$$-(v'_j(u_t) + 1)dD_t \leq 0,$$
since D is non-decreasing.

Hence, using again (3.32), we have

$$G_{t\wedge\tau} \leq v_I(u) + \int_0^{\tau\wedge t} \left(v_{I-N_s-1}\left(u_{s-} - h_s^{1,I-N_s}\right) - v_{I-N_s}\left(u_{s-}\right)\right)(dN_s - \lambda_{I-N_s}ds)$$
$$- \int_0^{\tau\wedge t} v_{I-N_s-1}\left(u_{s-} - h_s^{1,I-N_s}\right)\left(dH_s - \lambda_{I-N_s}(1-\theta_s^{I-N_s})ds\right). \qquad (3.33)$$

Now we have

$$\mathbb{E}\left[\int_0^{\tau\wedge t} \left|v_{I-N_s-1}\left(u_s - h_s^{1,I-N_s}\right) - v_{I-N_s}\left(u_s\right)\right| ds\right]$$
$$\leq \mathbb{E}\left[\int_0^{\tau\wedge t} \left|v_{I-N_s-1}\left(u_s - h_s^{1,I-N_s}\right) - v_{I-N_s-1}\left(u_s - b_{I-N_s}\right)\right| ds\right]$$
$$+ \mathbb{E}\left[\int_0^{\tau\wedge t} \left|v_{I-N_s-1}\left(u_s - b_{I-N_s}\right) - v_{I-N_s}\left(u_s\right)\right| ds\right]$$

Then, from (3.26), we know that for all j the function $u \longrightarrow v_j(u) - v_{j-1}(u-b_j)$ is decreasing. Moreover, for u large enough (namely $u \geq \gamma_j \vee (\gamma_{j-1} + b_j)$) we have

$$v_j(u) - v_{j-1}(u - b_j) = v_j(\gamma_j) + \gamma_j - v_{j-1}(\gamma_{j-1}) + \gamma_{j-1} - b_j,$$

which implies that for all j the function $u \longrightarrow v_j(u) - v_{j-1}(u - b_j)$ is bounded.

Moreover, we have

$$\mathbb{E}\left[\int_0^{\tau\wedge t} \left|v_{I-N_s-1}\left(u_s - h^{1,I-N_s}\right) - v_{I-N_s-1}\left(u_s - b_{I-N_s}\right)\right| ds\right]$$
$$\leq \mathbb{E}\left[\int_0^{\tau\wedge t} \left|h_s^{1,I-N_s} - b_{I-N_s}\right| \sup_{b_{I-N_s} < u \leq \gamma_{I-N_s}} \left|v'_{I-N_s}(u)\right| ds\right]$$
$$\leq C\left(1 + \mathbb{E}\left[\int_0^{\tau\wedge t} |u_s| ds\right]\right)$$
$$\leq C\left(1 + \mathbb{E}\left[\int_0^{\tau\wedge t} ue^{(r+2\lambda)s}ds\right]\right) < +\infty,$$

where $\lambda := \sup_{1\leq j\leq I} \lambda_j$, and where we used successively the fact that the derivative of the v_j can be extended to a continuous function on $[b_j, \gamma_j]$ which is therefore bounded on that compact, then the fact that by the limited liability condition (3.11) we have $h_t^1 \leq u_t$ and finally that conditionally on the fact that there are j loans left in the pool, the drift of u_t as given by (3.7) is

$$ru_t + \lambda_j\left(h_t^1 + (1-\theta_t)h_t^2\right) - \delta_t \leq ru_t + \lambda_j\left(h_t^1 + (1-\theta_t)(u_t - h_t^1)\right)$$
$$\leq ru_t + \lambda_j\left(u_t - b_{j-1} + (1-\theta_t)u_t\right)$$
$$\leq u_t(r + 2\lambda_j),$$

where we used the fact that h^1, b_j and λ_j are positive. Hence, u_t increases at a rate lower than $r + 2\lambda$.

Similarly, we have

$$
\mathbb{E}\left[\int_0^{\tau \wedge t} \left| v_{I-N_s-1}\left(u_{s^-} - h_s^{1,I-N_s}\right) \right| ds\right]
$$

$$
= \mathbb{E}\left[\int_0^{\tau \wedge t} \left| v_{I-N_s-1}\left(u_{s^-} - h_s^{1,I-N_s}\right) - v_{I-N_s-1}\left(u_{s^-} - h_s^{1,I-N_s} - h_s^{2,I-N_s}\right) \right| ds\right]
$$

$$
\leq \mathbb{E}\left[\int_0^{\tau \wedge t} \left| h_s^{2,I-N_s} \right| \sup_{b_{I-N_s} < u \leq \gamma_{I-N_s}} \left| v'_{I-N_s-1}(u) \right| ds\right]
$$

$$
\leq \mathbb{E}\left[\int_0^{\tau \wedge t} \left| u_s \right| \sup_{b_{I-N_s} < u \leq \gamma_{I-N_s}} \left| v'_{I-N_s-1}(u) \right| ds\right] < +\infty.
$$

Taking expectations in (3.33), we therefore obtain

$$
v_I(u_0) \geq \mathbb{E}\left[\int_0^\tau \left((I - N_s)\mu - \delta_s\right) ds\right] + \mathbb{E}\left[1_{t<\tau} \left(\int_t^\tau \left(\delta_s - (I - N_s)\mu\right) ds + v_{I-N_t}(u_t)\right)\right]
$$

$$
= \mathbb{E}\left[\int_0^\tau \left((I - N_s)\mu - \delta_s\right) ds\right] + \mathbb{E}\left[1_{t<\tau}\mathbb{E}_t\left[\int_t^\tau \left(\delta_s - (I - N_s)\mu\right) ds + v_{I-N_t}(u_t)\right]\right]
$$

$$
= \mathbb{E}\left[\int_0^\tau \left((I - N_s)\mu - \delta_s\right) ds\right] + \mathbb{E}\left[1_{t<\tau}\left(u_t + v_{I-N_t}(u_t) - \mathbb{E}_t\left[\int_t^\tau (I - N_s)\mu ds\right]\right)\right]
$$

$$
\geq \mathbb{E}\left[\int_0^\tau \left((I - N_s)\mu - \delta_s\right) ds\right] + \mathbb{E}\left[1_{t<\tau}\left(-I\mu\tau + u_t + v_{I-N_t}(u_t)\right)\right]. \tag{3.34}
$$

Then, we know that for all j the function $u \longrightarrow u + v_j(u)$ is increasing before γ_j and is constant for $u \geq \gamma_j$. It is therefore bounded and we have

$$
\left| -I\mu\tau + u_t + v_{I-N_t}(u_t) \right| \leq I\mu\tau + \sup_{1 \leq j \leq I} \left| \gamma_j + v_j(\gamma_j) \right| \leq C(1 + \tau),
$$

for some positive constant C. This quantity being integrable, we can apply the dominated convergence theorem in (3.34) and let t go to $+\infty$ to obtain

$$
v_I(u_0) \geq \mathbb{E}\left[\int_0^\tau \left((I - N_s)\mu - \delta_s\right) ds\right],
$$

which is the desired result. $\qquad \square$

4 What happens when $r = 0$?

In this section we treat our problem in the special case where the bank is as patient as the investors. We will see that in that case, the optimal contract leads to the first-best utility for the investors (but with fees paid continuously to the bank instead of a lump payment at time 0). Since most of the proofs follow exactly the same arguments as in the case $r > 0$, we will only sketch them. First, we give the analogue of Proposition 3.2 in that case.

Proposition 4.1. *Assume that $r = 0$.*

(i) *The ordinary differential equations (3.21), along with (3.20), have unique maximal solutions v_j for $j \geq 1$. The functions v_j are globally concave, differentiable everywhere except at b_j and twice differentiable everywhere except at b_j and $b_j + b_{j-1}$. The endogenous thresholds γ_j are uniquely determined by*

$$\gamma_j = \sum_{i=1}^{j} b_i. \tag{4.1}$$

(ii) *We also have*

$$v'_j(u) - v'_{j-1}(u - b_j) \leq 0, \ \text{for all } u \geq b_j. \tag{4.2}$$

Proof. (i) When $r = 0$, the solution of (3.21) for a given $\gamma \geq b_j$ is

$$v_j(u) = \frac{j\mu}{\lambda_j} + e^{\frac{u-\gamma}{b_j}} \left(v_{j-1}(\gamma - b_j) - b_j \right) + \int_u^\gamma \frac{e^{\frac{u-x}{b_j}}}{b_j} v_{j-1}(x - b_j) dx, \ b_j < u \leq \gamma$$

$$v_j(u) = \gamma - u + v_j(\gamma), \ u > \gamma.$$

Using the same arguments as in the proof of Proposition 3.2, it is easily proved that the choice of γ which leads to the maximum solution is

$$\gamma_j = \gamma_{j-1} + b_j.$$

Reasoning by induction, we can then prove similarly that the functions v_j verify all the desired properties. Moreover, since $\gamma_1 = b_1$, we obtain that

$$\gamma_j = \sum_{i=1}^{j} b_i.$$

(ii) We can prove that

$$v'_j(u) = \int_u^{\gamma_j} \frac{e^{\frac{u-x}{b_j}}}{b_j} \frac{dv_{j-1}}{du}(x - b_j) dx - e^{\frac{u-\gamma_j}{b_j}}, \ b_j < u \leq \gamma_j$$

$$\frac{dv_j}{du}(u) = -1, \ u > \gamma_j.$$

By the concavity of v_{j-1}, this implies that for $b_j < u \leq \gamma_j$

$$v'_j(u) - v'_{j-1}(u - b_j) \leq -e^{\frac{u-\gamma_j}{b_j}} \left(v'_{j-1}(u - b_j) + 1 \right) \leq 0.$$

Since (4.2) is clear when $u > \gamma_j$, this proves (ii). □

Thanks to Proposition 4.1, we have a concave solution of the HJB equation, then using the same techniques as in the case $r > 0$, we can verify that the optimal contract is given by

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Contract 4.1. (i) *Given size j, the pool remains in operation (i.e. there is no liquidation) with one less unit at any time there is a default in the range $\left[b_j + b_{j-1}, \gamma_j\right]$.*

(ii) *The flow of fees paid to the bank given j is $\delta_t^j = \lambda_j b_j$ as long as no default occurs.*

(iii) *Liquidation of the whole pool occurs with probability $\theta_t^j = \left(u_t - b_j\right)/b_{j-1}$ in the range $\left[b_j, b_j + b_{j-1}\right)$.*

Hence, when the bank starts with a reservation utility equal to γ_I (which is the only viable value for competitive investors, the contracts being on a take-it-or-leave-it basis), payments are never suspended since the penalty arising upon default brings the bank utility to the level γ_{I-1}. Besides, in that case it is never optimal to use the threat of stochastic liquidation. In that case, we also have

$$v_I(\gamma_I) = \frac{j\mu}{\lambda_j} - b_j + v_{j-1}(\gamma_{j-1}) = \frac{1}{\alpha_i}\left(\mu - \frac{B}{\varepsilon}\right) + v_{j-1}(\gamma_{j-1}) = \frac{I}{\alpha_I}\left(\mu - \frac{B}{\varepsilon}\right).$$

Therefore, the social value of the contract is

$$\gamma_I + v_I(\gamma_I) = \frac{I\mu}{\alpha_I},$$

which is exactly equal to $\mathbb{E}\left[\int_0^\tau \mu(I - N_t)dt\right]$, that is to say the social value which can be attained in the first-best. Hence, when the bank is no longer impatient, our contract leads to the same utility as in the first-best. This was to be expected, since there is no longer loss in utility due to the fact that the bank has to be penalized because of its impatience.

We also refer to [15] for heuristic results when other hypotheses of the model are relaxed.

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A Appendix

Proof. [Proof of Proposition 3.1] It is clear from the equation (3.16) that the right-derivative of v_1 at b_1 is equal to $\frac{\lambda_1 v_1(b_1) - \mu}{b_1(r+\lambda_1)}$. We therefore have to consider three cases.

- If $\lambda_1 v_1(b_1) > \mu$.

Then at least on a small interval on the right of b_1, we have $v_1' \geq 0$. Thus on this interval the equation becomes

$$(r + \lambda_1)\, u v_1'(u) + \mu - \lambda_1 v_1(u) = 0,$$

whose solution is given by

$$\widetilde{v}_1(u) := \left(v_1(b_1) - \frac{\mu}{\lambda_1} \right) \left(\frac{u}{b_1} \right)^{\frac{\lambda_1}{\lambda_1 + r}} + \frac{\mu}{\lambda_1}.$$

Since this function has a derivative which is always positive, this means that in that case $v_1' + 1 > 0$ and therefore δ is always equal to 0. Thus it follows that the investor's utility is equal to (see (3.2) when $k = 0$)

$$v_1(b_1) = \mathbb{E} \left[\int_0^{\tau^1} \mu ds \right] = \frac{\mu}{\lambda_1},$$

contradicting the fact that $\lambda_1 v_1(b_1) > \mu$. Hence this case is not possible.

- If $\lambda_1 v_1(b_1) = \mu$.

Then using (3.2) with $k = 0$, we obtain that $\delta = 0$ (since we assumed that $\delta \geq 0$). Plugging this in (3.1), we get that the bank utility is equal to 0 even if the loan has not defaulted, which contradicts the fact that it should remain above its minimum level b_1. Hence this case is not possible either.

- If $\lambda_1 v_1(b_1) < \mu$.

Then at least on a small interval on the right of b_1, we have $v_1' \leq 0$. Thus on this interval the variational inequality becomes

$$\min \left\{ -(ru + \lambda_1 b_1)\, v_1'(u) - \mu + \lambda_1 v_1(u), v_1'(u) + 1 \right\} = 0.$$

The solution of the first ODE in the system is given by

$$\widehat{v}_1(u) := \left(v_1(b_1) - \frac{\mu}{\lambda_1} \right) \left(\frac{ru + \lambda_1 b_1}{rb_1 + \lambda_1 b_1} \right)^{\frac{\lambda_1}{r}} + \frac{\mu}{\lambda_1}.$$

First, we consider the case $r \leq \lambda_1$. In that case the above function is concave for $u > b_1$, its derivative decreases to $-\infty$ and is therefore always negative. We will also verify next that we have

$$v_1'(b_1) \geq -1. \tag{A.1}$$

This implies that the solution v_1 is equal to $\widehat{v}_1$ until its derivative reaches the value -1 at some uniquely defined point γ_1. Thus, we have a solution on the interval $[b_1, \gamma_1]$. In that case we know that $\delta = 0$ for $u < \gamma_1$. In order to obtain the value of δ when $u = \gamma_1$, we return to the bank's utility dynamics given by (3.7)

$$du_t^0 = (ru_t^0 - \delta_t + \lambda_1(h_t^1 + (1 - \theta_t)h_t^2)dt, \text{ for } t < \tau^1.$$

Since $h^1 = b_1$, $h_2 = u - b_1$ and $\theta = 1$, we obtain

$$du_t^0 = (ru_t^0 - \delta_t + \lambda_1 b_1)dt, \text{ for } t < \tau^1.$$

Hence, if $u_0^0 < \gamma_1$ then $\delta = 0$ and thus the utility of the bank keeps on increasing until the default occurs or until the time t^* for which $u_{t^*}^0 = \gamma_1$. Then, δ_t should be chosen so that u_t^0 stays constant after that time t^*, that is to say

$$\delta_t = 1_{t=t^*}(r\gamma_1 + \lambda_1 b_1).$$

Indeed, if $\delta_{t^*} < r\gamma_1 + \lambda_1 b_1$ then u^0 keeps on increasing after t^* and therefore δ_t is equal to 0 except at t^*, and thus the utility of the bank given by (3.1) is 0, which contradicts the fact that it should stay above b_1. We obtain similarly a contradiction when $\delta_{t^*} > r\gamma_1 + \lambda_1 b_1$.

Now we want to calculate $v_1(b_1)$. First, in this case $u_0^0 = b_1$, and we therefore have after some calculations

$$t^* = \frac{1}{r}\ln\left(\frac{r\gamma_1 + \lambda_1 b_1}{rb_1 + \lambda_1 b_1}\right),$$

and thus by definition

$$\begin{aligned}
v_1(b_1) &= \frac{\mu}{\lambda_1} - (r\gamma_1 + \lambda_1 b_1)\mathbb{E}^{\mathbb{P}}\left[1_{t^*<\tau^1}(\tau^1 - t^*)\right] \\
&= \frac{\mu - (r\gamma_1 + \lambda_1 b_1)e^{-\lambda_1 t^*}}{\lambda_1} \\
&= \frac{\mu}{\lambda_1} - \frac{r\gamma_1 + \lambda_1 b_1}{\lambda_1}\left(\frac{rb_1 + \lambda_1 b_1}{r\gamma_1 + \lambda_1 b_1}\right)^{\frac{\lambda_1}{r}}.
\end{aligned}$$

Now recall that we have to verify that (A.1) holds. With the above value of $v_1(b_1)$, we obtain

$$v_1'(b_1) = -\left(\frac{rb_1 + \lambda_1 b_1}{r\gamma_1 + \lambda_1 b_1}\right)^{\frac{\lambda_1}{r}-1} \geq -1.$$

Hence, it remains to verify that we indeed have that u_0^0 calculated with (3.1) is equal to b_1.

We have

$$u_0^0 = (r\gamma_1 + \lambda_1 b_1)\mathbb{E}^{\mathbb{P}}\left[1_{t^* < \tau^1}\int_{t^*}^{\tau} e^{-rs}ds\right]$$

$$= \frac{r\gamma_1 + \lambda_1 b_1}{r}\mathbb{E}^{\mathbb{P}}\left[1_{t^* < \tau^1}\left(e^{-rt^*} - e^{-r\tau^1}\right)\right]$$

$$= \frac{r\gamma_1 + \lambda_1 b_1}{r + \lambda_1}e^{-(\lambda_1 + r)t^*}$$

$$= b_1\left(\frac{rb_1 + \lambda_1 b_1}{r\gamma_1 + \lambda_1 b_1}\right)^{\frac{\lambda_1}{r}}.$$

Thus, $u_0^0 = b_1$ if and only if we actually have $\gamma_1 = b_1$, which means that v_1 should be linear above b_1

$$v_1(u) = v_1(b_1) - u + b_1, \ u \geq b_1.$$

We now need to verify that

$$-(ru + \lambda_1 b_1)v_1'(u) - \mu + \lambda_1 v_1(u) \geq 0, \ u \geq b_1.$$

We have

$$-(ru + \lambda_1 b_1)v_1'(u) - \mu + \lambda_1 v_1(u) = r(u - b_1) \geq 0,$$

which shows that we indeed have found a solution of the variational inequality when $r \leq \lambda_1$.

Now, if $r > \lambda_1$, the function $\widehat{v}_1$ becomes convex and thus its derivative increases. Therefore, if $\widehat{v}_1'(b_1^+) > -1$, then δ is always equal to 0 and arguing as above this case is not possible. Hence $\widehat{v}_1'(b_1^+) = -1$, and we end up with the same solution as in the case $r \leq \lambda_1$.

Finally, we compute that

$$v_1'(b_1^-) - v_1'(b_1^+) = \frac{\mu - b_1(r + \lambda_1)}{\lambda_1 b_1} + 1 = \frac{\mu - rb_1}{\lambda_1 b_1} \geq \frac{\mu + B}{1 + \varepsilon} > 0,$$

by Assumption 2.2, we have $\mu - rb_1 \geq \frac{\mu + B}{1 + \varepsilon} > 0$, which implies that v_1 is concave.

$\square$

Proof. [Proof of Proposition 3.2(i)] We will show the result by induction.

- Initialization with $j = 2$

The solution of the ODE (3.21) for $j = 2$ and a given fixed value of $\gamma \geq b_2$ can be easily calculated and is given by

$$\widetilde{v}_2(u, \gamma) := (ru + \lambda_2 b_2)^{\frac{\lambda_2}{r}}\int_u^{\gamma}\frac{2\mu + \lambda_2 v_1(x - b_2)}{(rx + \lambda_2 b_2)^{\frac{\lambda_2}{r} + 1}}dx$$

$$+ \left(v_1(\gamma - b_2) + \frac{2\mu - (r\gamma + \lambda_2 b_2)}{\lambda_2}\right)\left(\frac{ru + \lambda_2 b_2}{r\gamma + \lambda_2 b_2}\right)^{\frac{\lambda_2}{r}}, \ b_2 < u \leq \gamma,$$

and $\widetilde{v}_2(u, \gamma) = \gamma - u + v_2(\gamma)$ for $u > \gamma$.

Now since we have shown that v_1 is everywhere twice differentiable except at b_1, we have for every $\gamma \neq b_1 + b_2$ and every $b_2 < u \leq \gamma$

$$\frac{\partial \widetilde{v}_2}{\partial \gamma}(u, \gamma) = \left(v_1'(\gamma - b_2) + 1 - \frac{r}{\lambda_2} \right) \left(\left(\frac{ru + \lambda_2 b_2}{r\gamma + \lambda_2 b_2} \right)^{\frac{\lambda_2}{r}} 1_{u \leq \gamma} + 1_{u > \gamma} \right).$$

Thus, the above expression always has the sign of $v_1'(\gamma - b_2) + 1 - \frac{r}{\lambda_2}$, that is to say that it is positive for $\gamma < b_1 + b_2$ and negative for $\gamma > b_1 + b_2$. Hence, we clearly have for all $b_2 < u$

$$\sup_{\gamma \geq b_2} \widetilde{v}_2(u, \gamma) = \widetilde{v}_2(u, b_1 + b_2),$$

which means that the maximal solution of (3.21) for $j = 2$ corresponds to the choice $\gamma_2 = b_1 + b_2$, which also happens to correspond to the unique solution of

$$\frac{r}{\lambda_2} - 1 \in \partial v_1(\gamma_2 - b_1).$$

Then, after some calculations, we obtain that for all $b_2 < u < b_1 + b_2$

$$v_2''(u) = - \left(\lambda_2 - r + \lambda_2 \frac{\overline{v}_1}{b_1} \right) \frac{(ru + \lambda_2 b_2)^{\frac{\lambda_2}{r} - 1}}{(r(b_1 + b_2) + \lambda_2 b_2)^{\frac{\lambda_2}{r}}} \leq 0,$$

because of (3.22).

Hence, since v_2, is linear on $[b_1 + b_2, +\infty)$ and is differentiable at $b_1 + b_2$, it is concave on $(b_2, +\infty)$. Now if we consider the linear extrapolation of v_2 over $[0, b_1]$ by (3.20), we just need to verify that the left-derivative of v_2 at b_2 is less than its right-derivative to obtain the concavity of v_2 over $[0, +\infty]$. Taking the limit for $u \downarrow b_2$ in the equation (3.21), we obtain

$$v_2'(b_2^+) = \frac{\lambda_2 \overline{v}_2 - 2\mu}{b_2(r + \lambda_2)}.$$

This implies that

$$v_2'(b_2^-) - v_2'(b_2^+) = \frac{2\mu}{b_2 \lambda_2} + v_2'(b_2^+) \frac{r}{\lambda_2} \geq \frac{\mu \epsilon}{B} - \frac{r}{\lambda_2}.$$

Now recall Assumption 2.2, which implies that

$$\frac{r}{\lambda_j} < \frac{r}{\overline{\alpha}_j} \leq \frac{\mu \epsilon - B}{B} \frac{\epsilon}{1 + \epsilon} < \frac{\mu \epsilon}{B}$$

for any j so that $v_2'(b_2^-) - v_2'(b_2^+) \geq 0$.

- Heredity : $j \geq 3$

Let us now suppose that the maximal solution of (3.21) v_{j-1} has been constructed for some $j \geq 3$, that it is globally concave on $[0, +\infty)$, everywhere differentiable except at b_{j-1}, everywhere twice differentiable except at b_{j-1} and $b_{j-1} + b_{j-2}$, and that the corresponding $\gamma_{j-1} \geq b_{j-1} + b_{j-2}$. Let us now construct the maximal solution corresponding to j. Exactly as in the case $j = 2$, the solution of the ODE (3.21) and a given fixed value of $\gamma \geq b_j$ can be easily calculated and is given by

$$
\widetilde{v}_j(u, \gamma) := (ru + \lambda_j b_j)^{\frac{\lambda_j}{r}} \int_u^\gamma \frac{j\mu + \lambda_j v_{j-1}(x - b_j)}{(rx + \lambda_j b_j)^{\frac{\lambda_j}{r}+1}} dx
$$
$$
+ \left(v_{j-1}(\gamma - b_j) + \frac{j\mu - (r\gamma + \lambda_j b_j)}{\lambda_j} \right) \left(\frac{ru + \lambda_j b_j}{r\gamma + \lambda_j b_j} \right)^{\frac{\lambda_j}{r}}, \quad b_j < u \leq \gamma,
$$

and $\widetilde{v}_j(u, \gamma) = \gamma - u + v_j(\gamma)$ for $u > \gamma$.

Note also that from (3.21) it is clear that v_j is differentiable everywhere except at b_j, and twice differentiable everywhere except at b_j and $b_j + b_{j-1}$.

Now since we assumed that v_{j-1} is everywhere differentiable except at b_{j-1}, we have for every $\gamma \neq b_{j-1} + b_j$ and every $b_j < u \leq \gamma$

$$
\frac{\partial \widetilde{v}_j}{\partial \gamma}(u, \gamma) = \left(v'_{j-1}(\gamma - b_j) + 1 - \frac{r}{\lambda_j} \right) \left(\left(\frac{ru + \lambda_j b_j}{r\gamma + \lambda_j b_j} \right)^{\frac{\lambda_j}{r}} 1_{u \leq \gamma} + 1_{u > \gamma} \right).
$$

Thus, since v_{j-1} is concave and its derivative non-increasing, we can conclude as in the case $j = 2$ that the maximal solution is uniquely determined by the choice γ_j which corresponds to the solution of

$$
\frac{r}{\lambda_j} - 1 \in \partial v_{j-1}(\gamma_j - b_j).
$$

More precisely, using (3.22), we have only two cases. Either,

$$
v'_{j-1}(b_{j-1}^+) \leq \frac{r}{\lambda_j} - 1 \leq \frac{\overline{v}_{j-1}}{b_{j-1}},
$$

and $\gamma_j = b_{j-1} + b_j$, or

$$
\frac{r}{\lambda_j} - 1 < v'_{j-1}(b_{j-1}^+),
$$

and $b_{j-1} + b_j < \gamma_j \leq \gamma_{j-1} + b_j$.

Let us now study the concavity. We can differentiate twice the equation (3.21) on $(b_j, b_j + b_{j-1})$ since $v_{j-1}(u - b_j)$ is linear and thus twice differentiable on this open interval. We then obtain easily

$$
v''_j(u) = v''_j((b_j + b_{j-1})^-) \left(\frac{ru + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j} \right)^{\frac{\lambda_j}{r}-2}, \quad b_j < u < b_j + b_{j-1}. \tag{A.2}
$$

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There are then two cases. If $\gamma_j = b_j + b_{j-1}$, differentiating once (3.21) and then taking the limit $u \uparrow b_j + b_{j-1}$, we get

$$(r(b_j + b_{j-1}) + \lambda_j b_j)v_j''((b_j + b_{j-1})^-) = \lambda_j \left(\frac{r}{\lambda_j} - 1 - \frac{\overline{v}_{j-1}}{b_{j-1}} \right) \leq 0.$$

Since $v_j''(u) = 0$ for $u > b_j + b_{j_1}$, we have proved the concavity on $(b_j, +\infty)$.

Now if $\gamma_j > b_j + b_{j-1}$, differentiating once (3.21) and taking limits on both sides of $b_j + b_{j-1}$, we obtain

$$v_j''((b_j + b_{j-1})^+) - v_j''((b_j + b_{j-1})^-) = \frac{\lambda_j}{r(b_j + b_{j-1}) + \lambda_j b_j} \left(\frac{\overline{v}_{j-1}}{b_{j-1}} - v_{j-1}'(b_{j-1}^+) \right), \quad \text{(A.3)}$$

where the right-hand side is positive by the concavity of v_{j-1}.

Next, we differentiate twice (3.21) on $(b_j + b_{j-1}, \gamma_j]$. We obtain easily

$$v_j''(u) = \lambda_j (ru + \lambda_j b_j)^{\frac{\lambda_j}{r} - 2} \int_u^{\gamma_j} \frac{v_{j-1}''(x - b_j)}{(ru + \lambda_j b_j)^{\frac{\lambda_j}{r} - 1}} dx. \quad \text{(A.4)}$$

Note that we should normally distinguish between the cases $b_j + b_{j-1} + b_{j-2} \leq \gamma_j$ or not, since v_{j-1} is not twice differentiable at $b_{j-1} + b_{j-2}$. However, since we know that v_j is twice differentiable at $b_j + b_{j-1} + b_{j-2}$, this actually does not change the result. Since v_{j-1} is concave, (A.4) implies that v_j is concave on $(b_j + b_{j-1}, +\infty)$. Then with (A.3) we obtain that the left second derivative of v_j at $b_j + b_{j-1}$ is negative, which, thanks to (A.2) shows finally the concavity on $(b_j, +\infty)$.

Finally, it remains to show that $v_j'(b_j^+) \leq \frac{\overline{v}_j}{b_j}$. We take the limit for $u \downarrow b_j$ in the equation (3.21), we obtain

$$v_j'(b_j^+) = \frac{\lambda_j \overline{v}_j - j\mu}{b_j(r + \lambda_j)}.$$

Since $v_j' \geq -1$, this implies that

$$v_j'(b_j^-) - v_j'(b_j^+) = \frac{j\mu}{b_j \lambda_j} + v_j'(b_j^+)\frac{r}{\lambda_j} \geq \frac{\mu\epsilon}{B} - \frac{r}{\lambda_j},$$

which has already been shown to be positive under Assumption 2.2. Hence v_j is concave on $[0, +\infty)$. $\qquad \square$

Proof. [Proof of Proposition 3.2(ii)] First of all, by the properties of the function ψ_1 recalled in Remark 3.3, it is clear that we can always find a λ_j such that (3.25) is satisfied. Then, if for a fixed $j \geq 2$ we have $v_{j-1}'(b_{j-1}^+) \leq 0$, by differentiating (3.21), we immediately have for $u > b_j$ and $u \neq b_j + b_{j-1}$

$$\lambda_j \left(v_j'(u) - v_{j-1}'(u - b_j) \right) = (ru + \lambda_j b_j)v_j''(u) + rv_j'(u). \quad \text{(A.5)}$$

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Since we have proved in (i) that the v_j are concave, it is clear that if $v'_{j-1}(b^+_{j-1}) \leq 0$, the right-hand side above is negative. Then by left and right continuity of v'_{j-1} at b_{j-1}, the result extends to $u = b_j + b_{j-1}$. Hence the desired property (3.26). In particular, this proves the result for $j = 2$ since $v'_1(b^+_1) = -1$.

Note also that the property (3.26) clearly holds for v_j when $u > \gamma_j$. Indeed, we have

$$v'_j = -1$$

and we know that the derivative of v_{j-1} is always greater than -1.

Let us now show the rest of the result by induction. Since (3.26) is true for $j = 2$, let us fix a $j \geq 3$ and assume that

$$v'_{j-1}(u) - v'_{j-2}(u - b_{j-1}) \leq 0, \ u > b_{j-1}. \tag{A.6}$$

Now if $v'_{j-1}(b^+_{j-1}) \leq 0$, we already know that the property 3.26 is true for v_j, so we will assume that $v'_{j-1}(b^+_{j-1}) > 0$. Moreover, by our remark above, we know that (3.26) holds true for v_j when $u > \gamma_j$. Let us then first prove that (3.26) for v_j when $u > b_j + b_{j-1}$. If $\gamma_j = b_j + b_{j-1}$, there is nothing to do. Otherwise, we have using successively (A.5) and (A.4)

$$\lambda_j \left(v'_j(u) - v'_{j-1}(u - b_j) \right) = (ru + \lambda_j b_j)v''_j(u) + rv'_j(u)$$
$$= (ru + \lambda_j b_j)^{\frac{\lambda_j}{r} - 1} \int_u^{\gamma_j} \frac{\lambda_j v''_{j-1}(x - b_j)}{(rx + \lambda_j b_j)^{\frac{\lambda_j}{r} - 1}} dx + rv'_j(u). \tag{A.7}$$

Now if we differentiate (3.21) and solve the corresponding ODE for v'_j, we obtain

$$v'_j(u) = (ru + \lambda_j b_j)^{\frac{\lambda_j}{r} - 1} \int_u^{\gamma_j} \frac{\lambda_j v'_{j-1}(x - b_j)}{(rx + \lambda_j b_j)^{\frac{\lambda_j}{r}}} dv - \left(\frac{ru + \lambda_j b_j}{r\gamma_j + \lambda_j b_j} \right)^{\frac{\lambda_j}{r} - 1}. \tag{A.8}$$

Using (A.8) in (A.7), we obtain for $u > b_j + b_{j-1}$

$$\lambda_j \left(v'_j(u) - v'_{j-1}(u - b_j) \right)$$
$$= \lambda_j (ru + \lambda_j b_j)^{\frac{\lambda_j}{r} - 1} \int_u^{\gamma_j} \frac{(rx + \lambda_j b_j)v''_{j-1}(x - b_j) + rv'_{j-1}(x - b_j)}{(rx + \lambda_j b_j)^{\frac{\lambda_j}{r}}} dv$$
$$- r \left(\frac{ru + \lambda_j b_j}{r\gamma_j + \lambda_j b_j} \right)^{\frac{\lambda_j}{r} - 1}. \tag{A.9}$$

Then we have for all $x \geq u > b_j + b_{j-1}$ and $x \neq b_j + b_{j-1} + b_{j-2}$

$$
\begin{aligned}
(rx + \lambda_j b_j) v''_{j-1}(x - b_j) + r v'_{j-1}(x - b_j) &= (r(x - b_j) + \lambda_{j-1} b_{j-1}) v''_{j-1}(x - b_j) \\
&\quad + (\lambda_j b_j - \lambda_{j-1} b_{j-1} + r b_j)\, v''_{j-1}(x - b_j) \\
&\quad + r v'_{j-1}(x - b_j) \\
&= \lambda_{j-1} \left(v'_{j-1}(x - b_j) - v'_{j-2}(x - b_j - b_{j-1}) \right) \\
&\quad + (\lambda_j b_j - \lambda_{j-1} b_{j-1} + r b_j)\, v''_{j-1}(x - b_j) \\
&\leq (\lambda_j b_j - \lambda_{j-1} b_{j-1} + r b_j)\, v''_{j-1}(x - b_j),
\end{aligned}
$$

where we used the induction hypothesis (A.6) in the last inequality.

Since v_{j-1} is concave, the sign of the right-hand side above is given by the sign of

$$
\lambda_j b_j - \lambda_{j-1} b_{j-1} + r b_j = \frac{JB}{\varepsilon} - \frac{(J-1)B}{\varepsilon} + r b_j = \frac{B}{\varepsilon} + r b_j \geq 0.
$$

Reporting this in (A.9) implies

$$
v'_j(u) - v'_{j-1}(u - b_j) \leq 0, \ \ u > b_j + b_{j-1}.
$$

It remains to prove (3.26) when $b_j < u < b_j + b_{j-1}$. In that case, (3.26) can be written

$$
v'_j(u) - \frac{\overline{v}_{j-1}}{b_{j-1}} \leq 0, \ \ b_j < u < b_j + b_{j-1},
$$

which is equivalent by concavity of v_j to

$$
v'_j(b_j^+) - \frac{\overline{v}_{j-1}}{b_{j-1}} \leq 0.
$$

Now using (A.8), we also have

$$
\begin{aligned}
v_j'(b_j^+) &= (ru + \lambda_j b_j)^{\frac{\lambda_j}{r}-1} \int_{b_j}^{b_j+b_{j-1}} \frac{\lambda_j \frac{\overline{v}_{j-1}}{b_{j-1}}}{(rx + \lambda_j b_j)^{\frac{\lambda_j}{r}}} dv \\
&\quad + v_{j-1}'(b_j + b_{j-1}) \left(\frac{ru + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j} \right)^{\frac{\lambda_j}{r}-1} \\
&= \frac{\overline{v}_{j-1}}{b_{j-1}} \frac{\lambda_j}{\lambda_j - r} \left(1 - \left(\frac{rb_j + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j} \right)^{\frac{\lambda_j}{r}-1} \right) \\
&\quad + v_{j-1}'(b_j + b_{j-1}) \left(\frac{ru + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j} \right)^{\frac{\lambda_j}{r}-1} \\
&\leq \frac{\overline{v}_{j-1}}{b_{j-1}} \frac{\lambda_j}{\lambda_j - r} \left(1 - \left(\frac{rb_j + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j} \right)^{\frac{\lambda_j}{r}-1} \right) \\
&\quad + v_{j-1}'(b_{j-1}^+) \left(\frac{ru + \lambda_j b_j}{r(b_j + b_{j-1}) + \lambda_j b_j} \right)^{\frac{\lambda_j}{r}-1} \\
&= \phi_{\frac{b_{j-1}}{b_j}} \left(\frac{r}{\lambda_j} \right) \frac{\overline{v}_{j-1}}{b_{j-1}} \left(\frac{\phi_{\frac{b_{j-1}}{b_j}} \left(\frac{r}{\lambda_j} \right) - 1}{\phi_{\frac{b_{j-1}}{b_j}} \left(\frac{r}{\lambda_j} \right)(x - 1)} + \frac{v_{j-1}'(b_{j-1}^+)}{\frac{\overline{v}_{j-1}}{b_{j-1}}} \right),
\end{aligned}
$$

which implies

$$
v_j'(b_j^+) - \frac{\overline{v}_{j-1}}{b_{j-1}} \leq \phi_{\frac{b_{j-1}}{b_j}} \left(\frac{r}{\lambda_j} \right) \frac{\overline{v}_{j-1}}{b_{j-1}} \left(\frac{v_{j-1}'(b_{j-1}^+)}{\frac{\overline{v}_{j-1}}{b_{j-1}}} - \psi_{\frac{b_{j-1}}{b_j}} \left(\frac{r}{\lambda_j} \right) \right).
$$

By Assumption 2.3, we know that $b_j \geq b_{j-1}$, hence with (3.25) and what we recalled earlier about the functions ψ_β in Remark 3.3, we have

$$
\frac{\overline{v}_{j-1}}{b_{j-1}} \leq \psi \left(\frac{r}{\lambda_j} \right) \leq \psi_{\frac{b_{j-1}}{b_j}} \left(\frac{r}{\lambda_j} \right),
$$

which implies the desired property and ends finally the proof.

$\square$

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