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Expositions bilatérales et risque systémique de solvabilité

Résumé : En utilisant une structure des états financiers des banques qui tient compte de leurs expositions bilatérales en termes d'actions et de prêts, on développe un modèle structurel de faillite. Ce modèle permet de distinguer les facteurs exogènes et endogènes dont dépend la faillite. On prouve l'existence et l'unicité de l'équilibre de liquidation, on étudie les conséquences des chocs exogènes sur le système bancaire, et on mesure le phénomène de contagion. On illustre l'usage de cette approche en l'appliquant au système bancaire français.

Mots-clés : Contagion, Risque Systémique, Solvabilité, Compensation, Équilibre de Liquidation, Impulsion-Réponse, Modèle de la Valeur-de-la-Firme.

Code JEL : G21, G28, G18, G33

Bilateral Exposures and Systemic Solvency Risk

Abstract: By introducing a structure of the balance sheets of the banks, which takes into account their bilateral exposures in terms of stocks or lendings, we get a structural model for default analysis. This model allows distinguishing the exogenous and endogenous default dependence. We prove the existence and uniqueness of the liquidation equilibrium, we study the consequences of exogenous shocks on the banking system and we measure contagion phenomena. This approach is illustrated by an application to the French banking system.

Keywords: Contagion, Systemic Risk, Solvency, Clearing, Liquidation Equilibrium, Impulse Response, Value-of-the Firm Model.

JEL Classification: G21, G28, G18, G33

1 Introduction

Following the crisis, the measurement of risk of financial institutions has become a critical question. How far will the value of a particular bank fall after an exogenous shock? How large would a shock on asset values have to be in order for a particular bank to go bankrupt ? Consider a set of financial institutions. If these institutions are not linked, then measuring the exposure to a change in the prices of several assets would be straightforward, given information on the institution's balance sheet only. For instance, if a bank owns 100 millions in a specific corporate stock and if the market price of this stock drops by 2%, then the bank's assets drops from 100 to 98 millions.
The problem, as revealed emphatically in the financial crisis, is that financial institutions are linked : each bank has ownership of a set of exogenous assets and as well as shares of other banks equity and loans. Therefore, measuring the risk of a financial institution needs to take into account the interconnections between banks' balance sheets, i.e. to find a consistent set of balance sheet values prior to the shock, and a consistent set of values afterwards.
A main deficiency of the regulations and practices before the recent financial crisis is the stand-alone computation of risk measures, i.e. the evaluation of risk made independently for the different financial institutions, followed by a crude aggregation to deduce the magnitude of the global risk. This practice concerns the assets themselves : for instance, the ratings of sovereign bonds (resp. credits) are determined separately for the different countries (resp. borrowers), and are not really informative on the risk of a portfolio of such bonds (resp. credits). Loosely speaking a portfolio of AAA bonds might be as risky as a portfolio of AA bonds, if the AAA bonds are positively dependent. Similarly, the required capitals in Basel 2 were computed bank by bank, without taking into account the dependence between the risks of these institutions. Even if this practice is not the cause of the financial crisis, it participated in its development. Since they were jointly exposed to an exogenous adverse shock, the banks had to increase their required capital simultaneously, and, thus, they had an important demand for cash, or riskfree asset. As a consequence of this need for liquidity, they tried to sell quickly the stocks they possessed, which implied in two days a significant drop in market stock prices.

The post-crisis regulation (Basel 3, Financial Stability Board) highlights the importance of risk dependences and consider financial institutions (banks and insurance companies) as parts of a system. They focus on the risk of the system (systemic risk) and the role of each institution in this systemic risk.

For a given system, like the set of European banks, say, there exist two reasons for a joint increase of risks for a large number of institutions, that are common exogenous adverse shocks and contagion.
i) First, there can exist shocks on a factor exogenous to the system. For instance, the increase of a prime rate will have an effect on the monthly payment for adjustable rate mortgages, will imply default clustering for individual mortgages and will diminish the results of all institutions having an important quantity of such mortgages, or associated mortgage backed securities, in their balance sheets. The default on a sovereign bond is another example of an exogenous shock with joint effects on the risk of the institutions.
ii) Contagion phenomena may arise in a second step and can amplify significantly the effect of exogenous shocks. They are due to the connection between the institutions through the structure of their balance sheets. For instance, a bank failure will have an impact on the institutions holding loans, bonds, stocks of this bank. In extreme cases, this may imply the failure of other institutions and so on. These contagion phenomena and chains of failures (the so-called domino effect of solvency) can result from an exogenous shock specific to an institution, such as a management error or a fraud, not necessarily from a shock on a common risk factor.

There exist two streams of literature on risk dependences, depending on the kind of available data.

i) Some analyses are based on the values of the institutions. These values can be deduced from their balance sheets, possibly disaggregated by class of assets, or from their capitalizations if they are quoted on a stock market. However, these balance sheets give no information on the existing contagion channels. This explains why it is difficult with such reduced form approaches to disentangle the exogenous and contagion effects. Systemic risk measures such as the CoVaR [Adrian, Brunnermeier (2008)], the Marginal Expected Shortfall (MES) [Acharya et alii (2010), Brownlees, Engle (2011), Acharya, Engle, Richardson (2012)], the Euler allocations [see Gourieroux, Monfort (2011) for a detailed discussion], are examples of reduced form measures unable to identify the two components of risk dependences.
As usual there exist two solutions to an identification problem. First, we

can constrain the model by introducing identification restrictions. This approach is followed in a static framework by Rosch, Winterfeld (2008), who set ex-ante to 20 % the number of contaminating firms. Another identification method is used in a dynamic framework by Gagliardini, Gourieroux (2012), Darolles, Gagliardini, Gourieroux (2012). Intuitively, simultaneity effects can be disentangled from lagged exogenous factor effects, interpreted as contagion. However, such identification restrictions are always rather ad hoc.

ii) An alternative approach is based on more informative data sets. In our framework, we need balance sheets disaggregated by class of assets and counterparties, not by class of assets only. Equivalently, we need the exposures of each bank for each class of asset and each counterparty. This type of data might become available soon due to the reporting by banks and insurance companies required by the new regulations on financial stability. They were not available in the past, except for specific segments of bank interlending, corresponding to some payment systems. For instance Humphrey (1986) uses data from the Clearinghouse Interbank Payments System and Furfine (2003) from Fedwire, the Federal Reserve's large value transfer system [see also McAndrews, Wasilyev (1995), Angelini, Maresca, Russo (1996), Elsinger, Lehar, Summer (2004), (2006)]. Other papers try to construct the missing data by using the knowledge on marginal exposures and by looking for the least favorable bilateral exposures [see e.g. Maurer, Sheldon (1998), Upper, Worms (2004), Upper (2011), Moussa (2011)]. This methodology is largely used in the central banks [see e.g. Wells (2004), Degryse, Nguyen (2007), Mistrulli (2007), Toivanen (2009)]. However, this methodology is based on a rather ad-hoc statistical criterion, called information criterion[4], to reconstruct the missing exposures, and thus is without any financial or

[4] The usual method is the so-called "entropy minimization method". The underlying principle is that each institution seeks for diversifying its interbank interconnections.
Consider n banks whose total interbank assets and total interbank liabilities, respectively denoted a_i and l_i, for $i = 1, ..., n$, are known. The issue is to estimate the bilateral exposures $x_{i,j}$, $i = 1, ..., n$, $j = 1, ..., n$, considering that $\sum_j x_{i,j} = a_i$ and $\sum_i x_{i,j} = l_j$. Moreover, a usual assumption is to set $x_{i,i} = 0$ for $i = 1, ..., n$. In practice, this assumption is required to avoid that using the "entropy minimization method" leads to almost only self-exposures [see Upper (2011)].
Technically, let us denote $\tilde{X}$ the vector of size $n^2 - n$ containing the off-diagonal elements of the bilateral exposure matrix to be estimated, Z the vector of size $n^2 - n$ containing the off-diagonal elements of matrix $(a_i l_j)_{i,j=1,...,n}$, A a $2n \times (n^2 - n)$ matrix such that

risk interpretation.

The aim of our paper is to provide a complete theoretical analysis which distinguishes different types of contagion channels. We extend the seminal paper by Eisenberg, Noe (2001) [see also Demange (2011)] to channels involving stocks and lendings, instead of lendings only. Moreover, we allow for stakeholders, i.e. shareholders and debtholders, outside the system.
In Section 2, we describe the system and the balance sheets of the institutions when all institutions are alive. The interconnections between them can be summarized by matrices of exposures through stocks or lendings. Thus the framework requires that the counterparties of any financial assets were identified. Note that, for a large set of assets during the financial crisis, such as credit derivatives, it was often impossible to know who the counterparties were. The regular collection of this information is a main innovation of the new European regulation. Examples of exposure matrices are given for the French banking sector. In Section 3, we study the consequences on the system of an exogenous shock. This shock may imply defaults of some institutions and changes in the balance sheets of the surviving ones. We prove the existence and uniqueness of the equilibrium after the shock. We discuss how the equilibrium depends on the magnitude of the shock. In particular, we construct impulse response functions and we consider the case of stochastic shocks. In Section 4, we provide a methodology to disentangle the direct and contagion systemic effects on the liquidation equilibrium. Section 5 concludes. The proofs are gathered in Appendices.

2 Balance sheet and exposure

2.1 System and systemic risk

Before discussing systemic risk and its exogenous or contagion components, it is necessary to precisely define the system. The perimeter of the system

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$[A\tilde{X}]_i = \mathbb{1}_{i\in[1,n]}\sum_j x_{i,j} + \mathbb{1}_{i\in[n+1,2n]}\sum_j x_{j,i}$. The minimization of entropy is

$$\begin{array}{rl} \min_{\tilde{x}_1,\ldots,\tilde{x}_{n^2-n}} & \sum_{k=1}^{n^2-n}\tilde{x}_k ln(\frac{\tilde{x}_k}{\tilde{z}_k}) \\ s.t. & \tilde{X}\geq 0 \\ s.t. & A\tilde{X} = [a', l']' \end{array}$$

has to specify :

- the type of institutions : banks/insurance-companies/hedge funds...
- the activity zone : France/Europe/World.
- the type of balance sheet, including or not the off, the intraday payments and settlements...
- the numeraire : Dollar/Euro...
- the assets which are the possible channels for contagion : loans/stocks/ derivatives...
- the existing regulation : definition and management of failure, bankruptcy, unsolvency...

It is also necessary to say what are the changes of a given system considered as risky. It is possible to consider the structure of the system, for instance the number of institutions (possibly weighted by their values), then to analyze the (weighted) number of failures following a shock, and among these failures the part due to the initial shock and the part due to contagion.

However, in some situations, a defaulted bank can be merged with a safer one. This will modify the structure of this system, but not necessarily with a significant impact on the account of the system, obtained by consolidating the balance sheets of all institutions. Thus, it is necessary to choose between a global (consolidated) analysis of the system and an analysis of its structure.

2.2 Balance sheet

We consider below a simplified description of the balance sheet of the institutions $i = 1, \ldots, n$ in the system, and assume that the possible interconnections appear either through stocks, or debts[5]. The structure of the balance sheet of institution i is given in Table 1.

Y_i denotes the value of institution i, L_i the total value of its debt[6]. This debt value is equal to the nominal (contractual) value L_i^*, if the institution is

[5] We do not distinguish in the paper : bonds, loans and lendings. Thus we assume a uniform debt structure.

[6] Different terminologies are used in the literature , such as :

- for institution i : node...

Asset	Liability
$\pi_{i,1}Y_1$	L_i
$\vdots$	
$\pi_{i,n}Y_n$	
$\gamma_{i,1}L_1$	
$\vdots$	
$\gamma_{i,n}L_n$	
Ax_i	
A_i	L_i

Table 1: Balance Sheet of Bank i

alive, but can be strictly smaller if there is default. The debt includes the issued bonds as well as the deposits, the interlending and the over-the-counter loans. For expository purpose, we assume that the debt is homogeneous, that is, we do not distinguish the seniority levels[7], the maturities and the different degrees of liquidity of the debt. In particular, we focus on solvency constraints, not on liquidity constraints, contrary to a large part of the theoretical literature. Institution i holds a proportion $\pi_{i,j}$ of the total number of shares of institution j, and a given proportion $\gamma_{i,j}$ of its total debt[8]. Thus, we assume a proportional sharing among counterparties of the debt of institution j in case of default. Ax_i gathers the asset components, which are outside the system, i.e. that correspond to sovereign, corporates, households, or even banks (which do not belong to this system). In general $\sum_i \gamma_{i,j}$ is much smaller than 1, since a significant part of the debt is hold by outsiders, such as depositors.

At this stage we do not explain how the asset and liability components are balanced. Indeed, the values of Y and L depend on the situations of the

- for value of the firm Y_i : equity capital, equity, net equity...
- for debt L_i : liability, debt obligation...
- for external asset Ax_i : net worth, operating cash-flow...

[7]See Gouriéroux, Héam, Monfort (2012) for an extension to multiple seniority levels.
[8]These assets may be hold or correspond to an uncovered operation.

banks, that is, if they are in default, or alive.

When all institutions of the system are alive, the balance sheets are characterized by :

i) The exogenous asset values : $Ax = (Ax_1, \ldots, Ax_n)'$;

ii) The nominal values of the debt : $L^* = (L_1^*, \ldots, L_n^*)'$;

iii) The interconnections induced by stocks and debts, that are, the (n, n) exposure matrices $\Pi = (\pi_{i,j})$ and $\Gamma = (\gamma_{i,j})$, respectively.

In this situation, we get the standard accounting relationships:

$$\begin{cases} L_i &= L_i^*, \\ Y_i &= A_i - L_i, \qquad i = 1, \ldots, n, \\ &= \sum_{j=1}^{n} (\pi_{i,j} Y_j) + \sum_{j=1}^{n} (\gamma_{i,j} L_j^*) - L_i^* + Ax_i. \end{cases} \tag{2.1}$$

They provide the values of the firms when all institutions are alive :

$$Y = (Id - \Pi)^{-1} [(\Gamma - Id) L^* + Ax], \tag{2.2}$$

whenever $Id - \Pi$ is invertible (see Lemma A.1 in Appendix 2).

2.3 Exposure matrices

The banks and insurance companies regularly report detailed balance sheets intended to give shareholders, investors and Supervisory Authorities information on their activities. The information on the structure of the balance sheets can be obtained by an appropriate treatment of the Financial Report database established by the European Banking Authority. An example of templates is provided in Appendix 1. The banks (and insurance companies) have to report their connections when the amount is larger than 300 MEuros, or 10% if its total equity.

The public financial statements on balance sheets allow us to reallocate assets and liabilities by categories and counterparties. We can estimate for every quarter t since December 2007, the matrices of exposures Π_t, Γ_t, as well as the vector of contractual debts L_t^* and the vector of exogenous asset components Ax_t.

i) The exposure matrices depend on the selected perimeter. We provide in Table 2, the exposure matrices at date 12/31/2010. They concern a system of five large French banks. We have kept large firms in terms of total assets to get exposure matrices of reasonable dimension. In fact, the number of financial institutions can reach several hundreds of firms. There are about 1000 banking institutions, reduced to about 200 consolidated groups for France. However, the first dozen consolidated groups represent about 95% of the total asset value. The selected financial institutions include two banks quoted on the stock markets and two mutual saving banks. The fifth is mixed : it is originally a mutual saving bank with several regional mutual saving funds, but this bank has developed a publicly traded subsidiary. This subsidiary represents approximately 60% of the banking group. These banks are denoted A, B, C, D and E for confidentiality restrictions.
Let us first describe the exposure matrix for stocks Π. For pure mutual saving banks, the absence of stocks[9] implies zero columns. Only a part of the mixed bank is quoted so that the corresponding column is much lower than the two columns for quoted banks. The diagonal reports the part of the total equity of a group hold by itself.
The exposure matrix for loans Γ has non zero coefficients out of the diagonal: every bank is lending and borrowing from every other bank. This corresponds to a complete structure in Allen, Gale (2001) terminology. Since we consider consolidated groups, there is no self-lending and the diagonal elements of Γ are equal to zero.

These exposure matrices can vary significantly over time. This arises for instance after the supporting plans from governments and after new Basel 3 regulations introduced to reduce systemic risk and potential risk contagion.

ii) The knowledge of the exogenous asset components and their joint dynamics is also important, since it may be used to define the static/dynamic, deterministic/stochastic shocks of interest. We provide in Figures 1-2 the

[9]Rigourously, the two mutual saving banks have a quoted subsidiary. But since they are very small, we neglected them. Moreover, it might happen that one bank holds shares of a mutual saving banks. But this type of link is very uncommon.

Π (%)	Bank A	Bank B	Bank C	Bank D	Bank E
Bank A	0.00	0.00	0.23	0.23	0.14
Bank B	0.00	0.00	0.68	0.69	0.41
Bank C	0.00	0.00	0.39	0.71	0.42
Bank D	0.00	0.00	0.34	1.65	0.21
Bank E	0.00	0.00	0.30	0.31	0.30
Γ (%)	Bank A	Bank B	Bank C	Bank D	Bank E
Bank A	0.00	0.43	0.41	0.35	0.38
Bank B	0.90	0.00	1.22	1.04	1.14
Bank C	2.32	3.27	0.00	2.66	2.93
Bank D	0.45	0.63	0.61	0.00	0.57
Bank E	0.40	0.57	0.54	0.46	0.00

Table 2: Exposure Matrices for the Banking Sector (at 12/31/2010) [source: public financial statements]

evolutions of these exogenous components for one of the mutual saving bank and one of the commercial bank. The frequency is quarterly.
Two pricing methods coexist in the balance sheet reports, that are the marked-to-market approach mainly for trading activities and the contractual values for credit activities. A given asset has to remain evaluated using the same method during its holding time, but many exceptions exist [see e.g. WSJ (2011)]. These pricing methods differ from the liquidation values which are implicit in Merton's model. For expository purpose, the consequences of these different valorizations are not considered here.

The exogenous assets fall into four main categories. The trading category gathers all elements that are marked-to-market or corresponding to short term perspectives. The second category corresponds to the retail activity ; it mainly consists in mortgages. Corporate loans form a third category. The last one, sovereign, includes the assets whose counterparty is part of the public sector (governments, states, local,...).
We observe that the selected mutual saving bank and commercial bank have a different structure of activity portfolios. The mutual saving bank splits its assets similarly between retail and corporate activities. The commercial bank seems to favor trading activities over other activities.

Despite the differences in term of structure, a similar trend and cycle drive the evolution of the structure of balance sheets. We observe that the decrease of the part of asset dedicated to trading activity varies across banks.

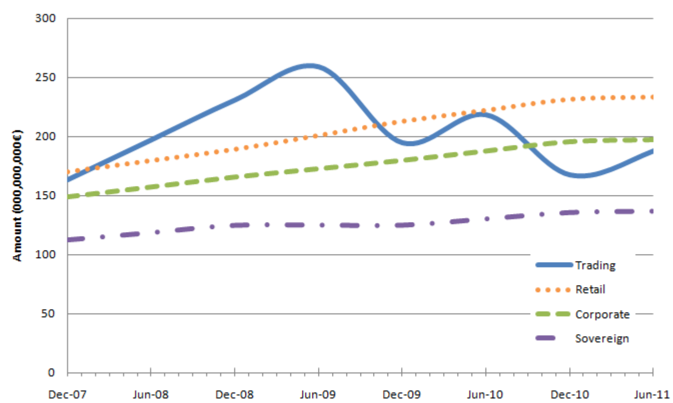

Figure 1: Exogenous Asset Components for a mutual saving bank [source: public financial statements]

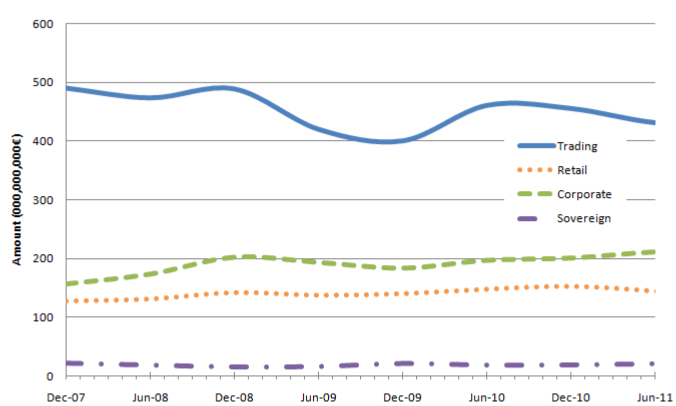

Figure 2: Exogenous Asset Components for a commercial bank [source: public financial statements]

3 Consequences of an exogenous shock

3.1 The liquidation equilibrium

Let us now consider an exogenous shock changing the initial exogenous asset value Ax^0 into Ax. This shock implies a change of the value of the firms and maybe the default of some institutions, which are no longer able to cover their nominal debt and will have zero value.

The values of the firms and the values of their debts after this shock are solutions of the system :

$$\begin{cases} Y_i = (A_i - L_i)^+, \\ L_i = \min(A_i, L_i^*). \end{cases}, i = 1, \ldots, n, \tag{3.1}$$

The first equation takes into account the possibility of default (when $A_i < L_i$) and the limited liability of shareholders [see Merton (1974)]. The second equation shows the seniority of debtholders with respect to shareholders. This implies an endogenous recovery rate equal to A_i/L_i^* in case of default[10]. It is easily seen that system (3.1) reduces to the standard Merton model for a system with a single firm and no self-holding of stocks, or bonds (see Appendix 7 for the analysis of equilibrium in the standard Merton's model).

System (3.1) can be written explicitly as :

$$\begin{cases} Y_i = \left[\sum_{j=1}^{n}(\pi_{ij}Y_j) + \sum_{j=1}^{n}(\gamma_{i,j}L_j) + Ax_i - L_i\right]^+, i = 1, \ldots, n, \\ L_i = \min\left[\sum_{j=1}^{n}(\pi_{i,j}Y_j) + \sum_{j=1}^{n}(\gamma_{i,j}L_j) + Ax_i, L_i^*\right]. \end{cases} \tag{3.2}$$

[10] A part of the literature assumes a constant exogenous recovery rate [see e.g. Furfine (2003), Upper, Worms (2004)], possibly set to zero [Cont et alii. (2010)]. In such a case, the piecewise affine mapping defining the equilibrium is no longer continuous and the existence and uniqueness of equilibrium is no longer ensured [Gouriéroux, Laffont, Monfort (1980)]. However, this assumption does not correspond to reality. For instance, the estimation of the recovery rates by Moody's are based on the value of the zero-coupon bonds of the firm one month after failure. Indeed, the bond market for a defaulted firm is still open and often shows a nonzero price of these bonds.

We get a $2n$-dimensional piecewise linear system, which can be solved to find a consistent set of values $Y = (Y_1, \dots, Y_n)'$, $L = (L_1, \dots, L_n)'$. As a by-product, the resolution of the system provides the institutions, which are in default : $\{Y_i = 0, L_i < L_i^*\}$, the set of institutions, which are still alive $\{Y_i > 0, L_i = L_i^*\}$, and the values of each asset business line of the alive institutions. In some particular cases, it has been proved that the consistent set of values Y, L can be interpreted as equilibrium values in an appropriate liquidation process managed by a centralized liquidator [see e.g. Demange (2011)]. This justifies the terminology liquidation equilibrium values used later on in the text.

The following Proposition is derived in Appendix 2 :

Proposition 1 : If $\pi_{i,j} \geq 0, \gamma_{i,j} \geq 0, \forall i,j, \sum_{i=1}^{n} \pi_{i,j} < 1, \forall j, \sum_{i=1}^{n} \gamma_{i,j} < 1, \forall j,$ the liquidation equilibrium Y, L exists and is unique for any choices of non-negative[11] Ax_i, L_i^*, $i = 1, ..., n$.

This equilibrium concerns the values of the institutions Y and the values of the debt L, and depends on the financial system $\mathcal{S} = \{\Pi, \Gamma, L^*, Ax\}$. Equivalently, if the numbers of shares are given and if there is a unique maturity of the debt, this is an equilibrium in the prices of stocks and digital credit default swap (CDS) written on the n institutions.
The result in Proposition 1 can be compared with the literature analyzing the existence and uniqueness of clearing repayment vector in the interlending market [see Eisenberg, Noe (2001), Demange (2011)]. In our notations,

[11] As noted in Demange (2011), there can exist situations with negative exogenous net worth Ax_i. In this case, the second equation in system (3.2) has to be written with an additional zero threshold as [see Elsinger et alii. (2006)]:

$$L_i = max\Big(\min \left[\sum_{j=1}^{n} (\pi_{i,j} Y_j) + \sum_{j=1}^{n} (\gamma_{i,j} L_j) + Ax_i, L_i^* \right] \ , \ 0),$$

and the regimes of default can now distinguish whether the recovery rate is equal to zero. However, this case arises if some debtors, such as depositors, are served before the banks in the system in case of default. This is an example of model with different seniority levels [see Gouriéroux, Héam, Monfort (2012)].

these papers assume no contagion by means of stocks, i.e. $\Pi = 0$, and an exposure matrix Γ with all columns summing up to 1. This explains why their proofs of existence and uniqueness rely on the interpretation in terms of graph structure of stochastic matrices. Proposition 1 completes their analysis in two respects : by introducing interconnection by means of stocks and by considering creditors outside the system. The proof is based on a necessary and sufficient condition for the invertibility of a piecewise linear function.

Let us illustrate the liquidation equilibrium as a function of the exogenous asset components for a system of two banks. For expository purpose, it is more appropriate to write the liquidation equilibrium conditions in terms of variables Y and $\Delta L = L^* - L$. Let us also denote $\Delta Ax = Ax - Ax^*$, where $Ax^* = (Id - \Gamma)L^*$. We get four possible regimes with the following liquidation equilibrium values :

Regime 1 : No default. We get : $Y = (Id - \Pi)^{-1}\Delta Ax, \Delta L = 0$,

and this regime occurs iff :

$$\Delta Ax \in (Id - \Pi)(I\!R^+)^2 \equiv \mathcal{C}_1.$$

Regime 2 : Joint default. We get : $Y = 0, \Delta L = (\Gamma - Id)^{-1}\Delta Ax$,

and this regime occurs iff :

$$\Delta Ax \in (\Gamma - Id)[0; L_1^*] \times [0; L_2^*] \equiv \mathcal{C}_2.$$

Regime 3 : Default of bank 1 only. We get : $Y_1 = 0, \Delta L_2 = 0$, and

$$\begin{pmatrix} \Delta L_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \gamma_{1,1} - 1 & -\pi_{1,2} \\ \gamma_{2,1} & 1 - \pi_{2,2} \end{pmatrix}^{-1} \Delta Ax.$$

The regime occurs if :

$$\Delta Ax \in \begin{pmatrix} \gamma_{1,1} - 1 & -\pi_{1,2} \\ \gamma_{2,1} & 1 - \pi_{2,2} \end{pmatrix} [0; L_1^*] \times I\!R^+ \equiv \mathcal{C}_3.$$

Regime 4 : Default of bank 2 only. We get : $\Delta L_1 = 0, Y_2 = 0$, and

$$\begin{pmatrix} Y_1 \\ \Delta L_2 \end{pmatrix} = \begin{pmatrix} 1-\pi_{1,1} & \gamma_{1,2} \\ & \\ -\pi_{2,1} & \gamma_{2,2}-1 \end{pmatrix}^{-1} \Delta Ax.$$

The regime occurs iff :

$$\Delta Ax \in \begin{pmatrix} 1-\pi_{1,1} & \gamma_{1,2} \\ & \\ -\pi_{2,1} & \gamma_{2,2}-1 \end{pmatrix} I\!R^+ \times [0; L_2^*] \equiv \mathcal{C}_4.$$

The sets $\mathcal{C}_j$, $j = 1, \ldots, 4$, are truncated positive cones. They are generated by the following pairs of vectors : $\mathcal{C}_1$ is generated by $(u_1, u_2), \mathcal{C}_4$ is generated by (u_4, u_1), $\mathcal{C}_2$ is generated by $(u_3, u_4), \mathcal{C}_3$ is generated by (u_2, u_3), where :

$$u_1 = \begin{pmatrix} 1-\pi_{1,1} \\ -\pi_{2,1} \end{pmatrix}, u_2 = \begin{pmatrix} -\pi_{1,2} \\ 1-\pi_{2,2} \end{pmatrix}, u_3 = \begin{pmatrix} \gamma_{1,1}-1 \\ \gamma_{2,1} \end{pmatrix}, u_4 = \begin{pmatrix} \gamma_{1,2} \\ \gamma_{2,2}-1 \end{pmatrix}. \tag{3.3}$$

The conditions in Proposition 1 ensure that these truncated cones do not overlap. The regimes are represented in Figure 3.

There is no default for both banks, if the exogenous asset components Ax_1 and Ax_2 are sufficiently large. We observe that there exists thresholds

$$\overline{Ax}_1 = Ax_1^* + \frac{1-\pi_{1,1}}{\pi_{2,1}} Ax_2^*, \quad \overline{Ax}_2 = Ax_2^* + \frac{1-\pi_{2,2}}{\pi_{1,2}} Ax_1^*,$$

such that if $Ax_i > \overline{Ax}_i$, institution i will not default whatever the exogenous asset component of the other institution.

The proof of the existence and uniqueness in the case of two banks is easy to understand. Indeed, there exists a unique equilibrium if the cones defining the regimes in Figure 3 do not overlap. Since the sign of $det(u, v)$, where u,v are vectors of $I\!R^2$, gives the direction of rotation from u to v, the condition is simply that the four determinants : $det(u_1, u_2)$, $det(u_2, u_3)$, $det(u_3, u_4)$ and $det(u_4, u_1)$ have the same sign. It is easily checked that all these determinants are strictly positive under the assumption on the exposure matrices given in Proposition 1.

The conditions on exposure matrices given in Proposition 1 are sufficient for the existence and uniqueness of the liquidation equilibrium. When they

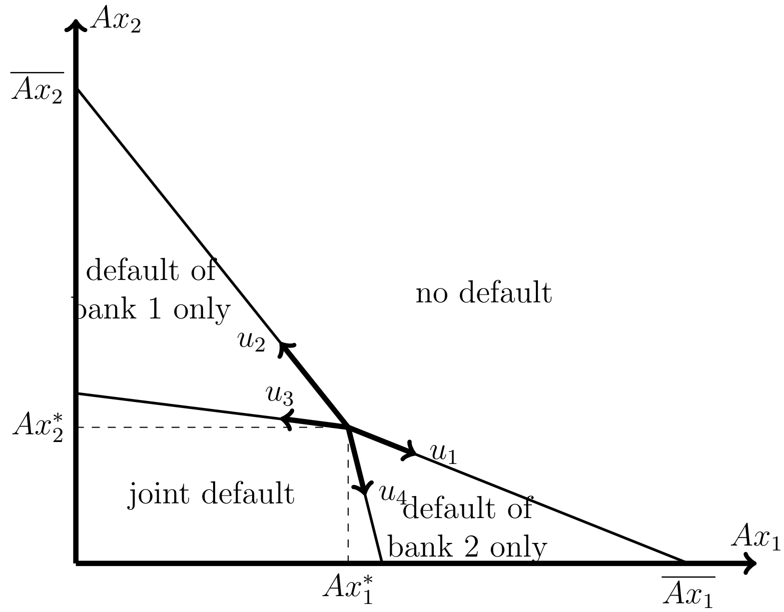

Figure 3: The Regimes of Default

are not satisfied, we do not have necessarily a unique liquidation equilibrium. For instance, we have a multiplicity of liquidation equilibria in Regime 1 if Π has a unitary eigenvalue, in particular if $\sum_{i=1}^{n} \pi_{i,j} = 1,\ j = 1,...,n$, since $(Id - \Pi)$ is not invertible. This is easily understood : if $\sum_{i=1}^{n} \pi_{i,j} = 1,\ j = 1,...,n$, the stock cross-holdings are so large, that we have in fact a unique group. The multiplicity of liquidation equilibria reveals that the consolidation step has not been well done[12].

3.2 Impulse Response Analysis and Stochastic Shock

i) Comparative statics

We can now discuss how the equilibrium responds to shocks on the exogenous asset components. We have the following monotonicity property :

[12]The counterexamples provided in Eisenberg, Noe (2001), Appendix 2, and Demange (2011), p11, are of the same type.

Proposition 2 : If $\pi_{i,j} \geq 0$, $\gamma_{i,j} \geq 0$, $\forall i, j$, $\sum_{i=1}^{n} \pi_{i,j} < 1$, $\forall j$, $\sum_{i=1}^{n} \gamma_{i,j} < 1$, $\forall j$,

the equilibrium values Y_i, L_i, $i = 1, ..., n$ are nondecreasing functions of the asset components Ax_j, $j = 1, ..., n$, for any given nominal debt and exposure matrices.

Proof : See the appendix on line.

This result was expected. An increase of the exogenous component decreases the default occurrence, increases the value of the firm, and also the recovery rate of any defaulting firm. It has been shown in Eisenberg, Noe (2001), that the debt level L is a componentwise concave function of Ax, when $\Pi = 0$ and $\sum_j \gamma_{i,j} = 1$, $\forall i$. This result is no longer valid when there is a feedback effect by means of stock cross-holdings.

ii) Impulse response analysis

Let us now consider an initial exogenous asset component Ax^0, a (multidimensional) direction of shocks $\beta = (\beta_1, ..., \beta_n)'$ and the new exogenous asset components defined by :

$$Ax(\delta) = Ax^0 + \delta\beta, \tag{3.4}$$

where δ, $\delta > 0$, is the magnitude of the deterministic shock. The impulse response[13] explains how the equilibrium values Y and L depend on δ, for given β and initial conditions.

As usual in the current regulation, the effects of the shocks are analyzed with crystallized, i.e. fixed, bilateral exposure matrices. From an economic point of view, this might be interpreted as the effect of an immediate not anticipated shock. From a practical point of view, it is difficult for the regulator to make reasonable assumptions about the reaction of the financial institutions to the different types of shocks, or to get reliable information on the future strategies of the institutions under stress. In a strand of literature, banks' reactions are stylized in a mechanical reaction in order to stabilized a target ratio [see e.g. Greenwood, Landier, Thesmar (2012), Cifuentes, Ferrucci, Shin (2005)]. However,these reactions are partly taken into account

[13] An impulse response describes the reaction of a system to a function of time, or some other independent variable. The latter interpretation is used in our static framework.

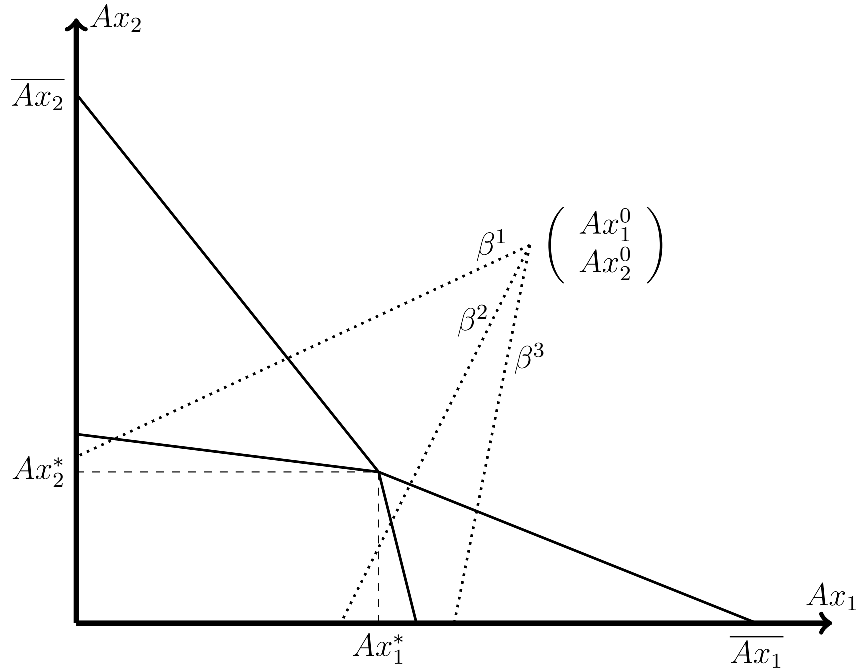

Figure 4: Directions of the Shocks

if these exercises are performed on a regular basis, monthly or quarterly, with updated bilateral exposure matrices[14].
A simple case is that of uniformly adverse shocks, when all components of the direction of the shocks are nonpositive : $\beta_i \leq 0$, $\forall i$. Indeed, by Proposition 2, we can apply the monotonicity property, and deduce a minimal value of δ : δ_1^*, say, for which we observe the first default, then a minimal value, δ_2^*, say, for which we observe the first two defaults, and so on. By studying the thresholds of magnitude δ of the shock that trigger default, we build the inverse impulse response. Central bankers call this approach "reverse stress test" [see e.g. BIS (2009), or FSA (2009)]. This is illustrated in Figure 4 for a system of two banks.

The initial situation corresponds to a banking system in a joint no default regime. A direction of shock $\beta = (\beta_1, \beta_2)'$, with $\beta_1 \leq 0, \beta_2 \leq 0$ defines a half-

[14]The histories of bilateral exposure matrices may be used to introduce a dynamic definition of shocks and to understand how the financial institutions adjust their strategies.

line with negative slope. This line can cross between 0 and 2 other regimes. For instance the directions β^1 and β^2 displayed on Figure 4 cross two other regimes, whereas direction β^3 crosses only one.

Figure 5 reports the impulse response functions with a direction β^2 for different characteristics of the equilibrium, that are the values of the exogenous asset components, the default indicators, the values of the firms and the values of the debts. The magnitude δ of the shock is on the x-axis. We set :

$$\Pi = \begin{pmatrix} 0.05 & 0.37 \\ 0.46 & 0.07 \end{pmatrix} \; ; \; \Gamma = \begin{pmatrix} 0.07 & 0.13 \\ 0.15 & 0.00 \end{pmatrix} \; ; \; Ax^0 = \begin{pmatrix} 2.9 \\ 2.2 \end{pmatrix} \; ; \; L^* = \begin{pmatrix} 2.5 \\ 2 \end{pmatrix}$$

$$\beta^2 = \begin{pmatrix} -0.003 \\ -0.005 \end{pmatrix}$$

The effects of the shocks on the external asset component are reported on the North-West panel. As the values of external asset decrease both for bank 1 and bank 2, the values of the banks plotted on South-West panel decrease and stop at 0, triggering first the default of bank 2, then the default of bank 1. When in default, the value of the bank remains constant equal to zero, but the value of its debt, that is the recovery rate, starts decreasing (South-East panel). The status of a bank (North-East panel) is 0, when it is alive and 1, otherwise : bank 1 defaults first at $\delta_1^* \approx 125$, then bank 2 defaults at $\delta_2^* \approx 250$.We also observe the convexity of the value of the bank and the concavity of the value of its debt corresponding to the call and put interpretations, respectively (see Appendix 4, d).
Some components of Ax may become negative when δ is too large. For this reason, we increase δ up to the first zero component of Ax.

iii) Stochastic shock

We can also consider a stochastic new situation Ax, and the associated stochastic shock $Ax - Ax^0$. This situation is characterized by the multivariate distribution of the vector of exogenous asset components. Then we can deduce explicitly the distribution of the equilibrium values Y and L. More precisely, let us introduce the regime indicator : $z = (z_1, ..., z_n)$, where $z_i = 1$, if bank i defaults, $z_i = 0$ otherwise. It is shown in Appendix 2 that regime z occurs iff :

$$\Delta Ax = Ax - Ax^* \in \mathcal{C}(z),$$

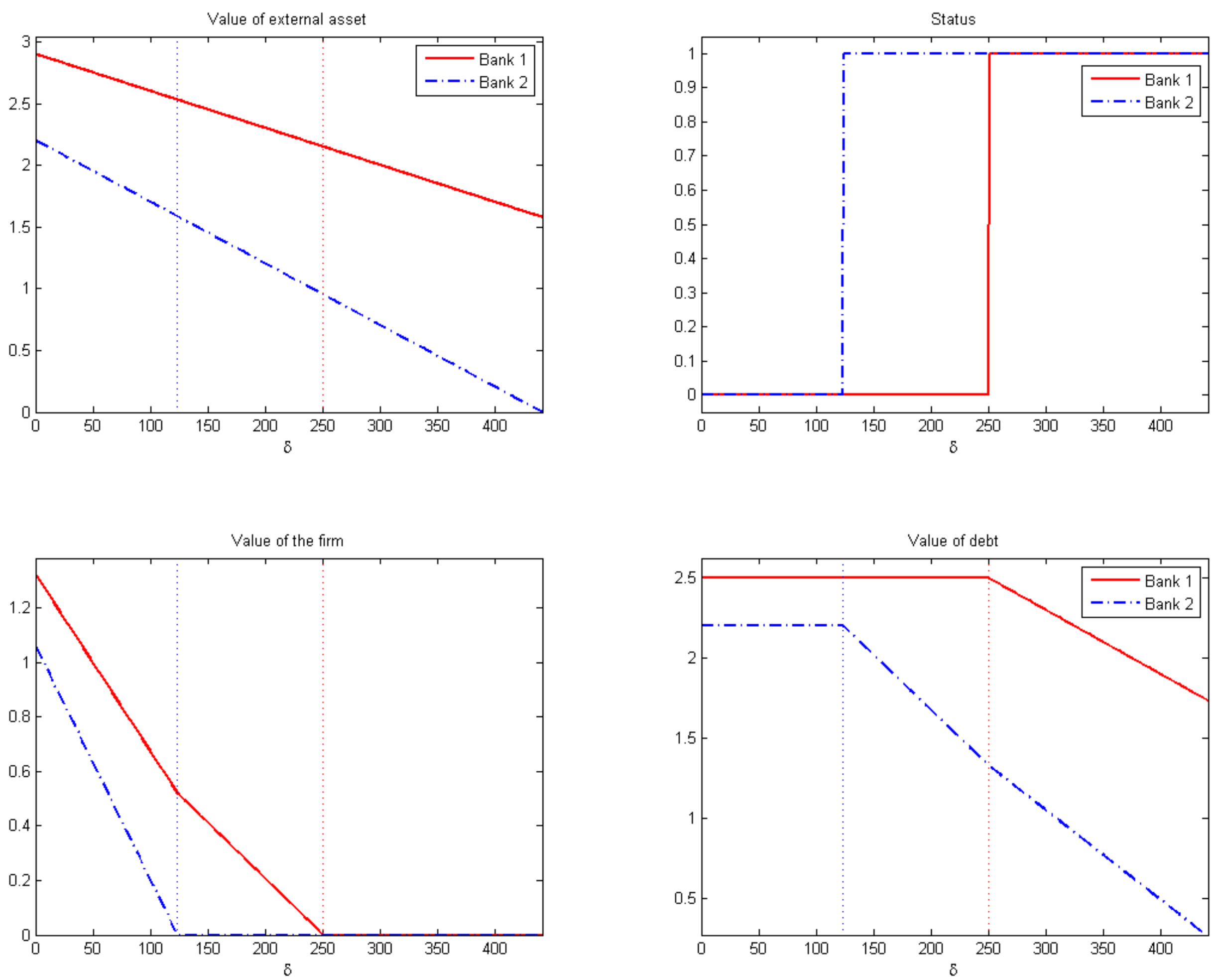

Figure 5: Impulse Response Functions

where $Ax^* = (Id - \Gamma)L^*$, $\mathcal{C}(z)$ is a truncated cone, function of Π, Γ and L^*, defined in Appendix 2. Therefore the probability to be in regime z is :

$$\mathbb{P}(\text{regime } z) = \mathbb{P}[Ax - Ax^* \in \mathcal{C}(z)].$$

Then it is easy to derive the conditional distribution of $(Y, L)' = X$, say, given the regime. Indeed, in regime z, we have :

$$Y_i = 0, \text{ if } z_i = 1, \quad L_i = L_i^*, \text{ if } z_i = 0.$$

Let us denote by X_z the n-dimensional vector obtained by stacking the value of Y_i for the banks such that $z_i = 0$, and the value of L_i, for the banks such that $z_i = 1$. It is proved in Appendix 2, that X_z is an invertible linear function of ΔAx in regime z :

$$X_z = B_z \Delta Ax,$$

where B_z is a matrix function of Π and Γ, whose expression is given in Appendix 2. Therefore, in regime z, the vector X_z has a n-dimensional distribution with density :

$$h(x_z) = \frac{1}{det(B_z)} f(B_z^{-1} x_z),$$

where f denotes the density of ΔAx.
To summarize, the joint distribution of (Y, L) is a mixture of $2n$-dimensional distributions, which are continuous for n coordinates and discrete for the other ones.
We get a complicated uncertainty, which is well illustrated by considering for instance the probability of default (PD) of a given bank. The probability of default of bank 1, say, is given by :

$$\begin{aligned}
PD_1 &= \mathbb{P}(z_1 = 1) \\
&= \sum_{z/z_1=1} \mathbb{P}(\text{regime } z) \\
&= \mathbb{P}(\text{regime } (1, 0, ..., 0)) + \sum_{i \neq 1} \mathbb{P}(\text{regime } z_1 = 1, z_i = 1, z_j = 0, j \neq 1, i) \\
&+ \sum_{i,j/i \neq j \neq 1} \mathbb{P}(\text{regime } z_1 = 1, z_i = 1, z_j = 1, z_k = 0, k \neq 1, i, j) \\
&+ ...
\end{aligned}$$

Thus the standard PD can be decomposed to highlight the number of banks, which are in default jointly with bank 1 :

$$PD_1 = PD_1(1) + PD_1(2) + ... + PD_1(n), \text{ say.} \tag{3.5}$$

Similarly, we may compute the probability of a joint default of two banks $PD_{1,2}$, say, if these banks are 1 and 2, and decompose it according to the total number of defaults in the system. Such a decomposition may be used to complete the standard analysis of default correlation.

4 Contagion measure

4.1 The standard analysis in a linear framework

Let us consider linear system (2.1), which can be rewritten :

$$Y = \Pi Y + (\Gamma - Id)L^* + Ax^0, \text{ say,} \tag{4.1}$$

and let us introduce a deterministic shock on the exogenous asset component :

$$Ax = Ax^0 + \delta\beta, \tag{4.2}$$

where β denotes the direction of the shock and δ its magnitude. The effect on the equilibrium values of the firms is :

$$\Delta Y = \delta(Id - \Pi)^{-1}\beta. \tag{4.3}$$

This shock is linear in δ and involves both a direct effect of the shock and a contagion effect. To disentangle these two components, we usually introduce a recursive version of model (4.1) :

$$Y_k = \Pi Y_{k-1} + (\Gamma - Id)L^* + Ax^0, \tag{4.4}$$

leading to the equilibrium solution (4.1), when k tends to infinity, assuming that matrix Π has eigenvalues with modulus strictly smaller than one. Then we compute the short term effect of the shock, equal to $\delta\beta$, and decompose the total effect as :

$$\Delta Y = \delta(Id - \Pi)^{-1}\beta = \underbrace{\delta\beta}_{\substack{\text{direct effect}\\ \text{of the exogenous}\\ \text{shock}}} + \underbrace{\delta(\sum_{j=1}^{\infty}\Pi^{j})\beta}_{\substack{\text{contagion}\\ \text{effect}}}. \quad (4.5)$$

In this linear framework, the direct and contagion effects are both linear in the direction β of the shock and its magnitude δ. In particular, they can easily be deduced from the shocks specific to each institution, since :
$\beta = \beta_1(1, 0, \dots, 0)' + \beta_2(0, 1, 0, \dots, 0)' + \dots + \beta_n(0, 0, \dots, 0, 1)'$.
Moreover, the two components in (4.5) can be obtained directly without specifying an underlying recursive process. Indeed, the direct effect is simply obtained by setting $\Pi = 0$ in formula (4.1), that is by canceling the contagion channel in terms of stocks. Note that the direct effect is independent of Γ and therefore can be also computed under $\Pi = \Gamma = 0$.

4.2 How to disentangle exogenous and contagion effects ?

Let us consider an initial financial system with exogenous asset components Ax^0, in which all institutions are alive. As noted earlier, the equilibrium values are :

$$Y^0 = (Id - \Pi)^{-1}\Big[(\Gamma - Id)L^* + Ax^0\Big]. \quad (4.6)$$

The equilibrium values with contagion, when $Ax = Ax^0 + \delta\beta \geq 0$, are the solutions of system (3.2). They will be denoted by : $Y(\mathcal{S}^0; \delta, \beta)$ and $L(\mathcal{S}^0; \delta, \beta)$, where $\mathcal{S}^0 = \{\Pi, \Gamma, L^*, Ax^0\}$ characterizes the financial system.
It is easy to suppress the contagion channel in our framework, that is to get $\Pi = \Gamma = 0$. Indeed, let us assume that, in the initial financial system $\mathcal{S}^0$, the institutions cash their stocks and bonds of the other institutions[15]. The balance sheet becomes :

[15]Implicitly we assume liquid markets for stocks and bonds. This assumption is consistent with the conditions of $\sum_j \pi_{i,j} < 1$ and $\sum_j \gamma_{i,j} < 1$, which means that a part of stocks and bonds issued by institutions are hold by external agents (households, corporates...).

Asset	Liability
$\widetilde{Ax_i^0}$	L_i

where $\widetilde{Ax_i^0} = \Pi_i Y^0 + \Gamma_i L^* + Ax_i^0 = Y_i^0 + L_i^*$. We have eliminated the contagions by setting $\Pi = 0$ and $\Gamma = 0$, while keeping the same value of the firm. Let us now apply the exogenous shock to this new financial system $\widetilde{\mathcal{S}^0} = \{0, 0, L^*, \widetilde{Ax^0}\}$. We get another equilibrium $Y(\widetilde{\mathcal{S}^0}; \delta, \beta)$ and $L(\widetilde{\mathcal{S}^0}; \delta, \beta)$, such that:

- institution i is alive if and only if : $Y_i^0 + \delta\beta_i > 0$,
- $\tilde{Y}_i = (Y_i^0 + \delta\beta_i)^+$,
- $\tilde{L}_i = min(\widetilde{Ax_i^0} + \delta\beta_i \ ; L_i^*)$.

By comparing the two liquidation equilibria associated with $\mathcal{S}^0$ and $\widetilde{\mathcal{S}^0}$, respectively, we get the effect of contagion. This approach can be applied to different aggregate measures of the final state of the system such as :

i) The number of non-defaulted banks :

$$N_0 = \sum_{i=1}^{n} \mathbb{1}_{Y_i>0} = \sum_{i=1}^{n} \mathbb{1}_{L_i - L_i^* = 0}, \tag{4.7}$$

where $\mathbb{1}_A$ denotes the indicator function of A.

ii) The total value of the banks :

$$\bar{Y} = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} Y_i \mathbb{1}_{Y_i>0}, \tag{4.8}$$

which is a criterion appropriate for shareholders.

iii) The total value of the debt :

$$\bar{L} = \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} L_i^* \mathbb{1}_{Y_i>0} + \sum_{i=1}^{n} L_i \mathbb{1}_{L_i<L_i^*}, \tag{4.9}$$

which is a criterion appropriate for bondholders

All these scalar measures of the global state of the banking system are nondecreasing functions of $Y_i, L_i, i = 1, \ldots, n$, and, therefore, also nondecreasing functions of the exogenous asset components Ax_i, by the monotonicity property established in Section 3.2. If we consider for instance the number of non-defaulted banks, we compute $N_0(\mathcal{S}_0; \delta, \beta)$, $N_0(\widetilde{\mathcal{S}^0}; \delta, \beta)$, which are decreasing functions of δ (if $\beta_i < 0$, $\forall i$), and decompose the total effect $N_0(\mathcal{S}_0; \delta, \beta)$ into the direct effect $N_0(\widetilde{\mathcal{S}^0}; \delta, \beta)$ and the contagion effect equal to the difference $N_0(\mathcal{S}_0; \delta, \beta) - N_0(\widetilde{\mathcal{S}^0}; \delta, \beta)$. The (absolute and per-cent) contagion effects depend on the initial configuration $\mathcal{S}^0$, but also on the direction and magnitude of the shock. This dependence is rather complex, and, as already noted, the contagion effect is not a linear function of β. Therefore, we cannot deduce the effect of a global shock from the effects of the specific shocks. Because of this dependence of the shock, we have to avoid relying only on:

- the ranking of Systematically Important Financial Institutions (the so-called SIFI's in the terminology of the Financial Stability Board),
- the distinction between "shock transmitters" and "shock absorbers" [Nier et alii (2007)],
- a definition of contagion measure based on a unique type of shock, such as the so-called market shock [Cont, Moussa, Santos (2010)].

The previous approach can also be applied by partly canceling contagion channels. For instance, we can set to zero, i.e. cash, all cross-holdings between the bank and insurance sectors to evaluate the effect of bancassurance business model on systemic risk. With respect to perimeter, we may also assess the effect of contagion through mutualization features or hedge fund industry. We may also cancel all links which do not involve a given institution i to focus on the contagion channel passing by this institution.

Let us illustrate the contagion effect in the case of two banks with nominal debts $L^* = (2, 3)'$ and values $Y^0 = (1, 1)'$, in all experiments below. We consider the following set of exposure matrices :

- Set 1 : $\Pi = \begin{pmatrix} 0 & 30\% \\ 0 & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,
- Set 2 : $\Pi = \begin{pmatrix} 0 & 0 \\ 30\% & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

- Set 3 : $\Pi = \begin{pmatrix} 0 & 30\% \\ 30\% & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,
- Set 4 : $\Pi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 & 30\% \\ 0 & 0 \end{pmatrix}$,
- Set 5 : $\Pi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 & 0 \\ 30\% & 0 \end{pmatrix}$,
- Set 6 : $\Pi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 & 30\% \\ 30\% & 0 \end{pmatrix}$,

For each set, both banks are assumed alive and the exogenous asset components are deduced by the formula : $Ax^0 = (Id - \Pi)Y^0 + (Id - \Gamma)L^*$. We consider a shock specific to bank 1, that is : $\beta = (-1, 0)'$. We report in Figures 6 and 7, the impulse response functions for the total debt, with their decompositions into direct and contagion effects.

The contagion effect depends on the system which is considered. It is always more pronounced on the total value of the debt. However, the effect of contagion is far to be clear in a general framework. It depends on the form of the exposure matrices, but also on the type of exogenous shock, deterministic or stochastic, of its direction and magnitude [see e.g. Dubecq, Gouriéroux (2012)]. As an illustration, let us consider a small stochastic shock such that $\delta = 1$ and β is such that the system stays in the no joint default regime. We know that there is no effect of debt exposure and that the decomposition of the effect of the shock on the value of the firm corresponds to equation (4.5). The per-cent contagion effect on the expected total value of the firm is:

$$1 - \frac{\mathbb{1}'\mathbb{E}(\beta)}{\mathbb{1}'(Id - \Pi)^{-1}\mathbb{E}(\beta)},$$

where $\mathbb{E}(\beta)$ denotes the expected multivariate shock and $\mathbb{1} = (1, \dots, 1)'$. In fact, the terminology per-cent contagion effect is misleading. Indeed, if this quantity is between 0 and 1 when the expected shocks are uniformly adverse $\mathbb{E}[\beta_i] \leq 0$, $i = 1, ..., n$, this is no longer the case in general. The contagion effect on the total value of the firm is more difficult to discuss for stochastic shock due to the dependence between direct and contagion effects [see also Darolles, Gagliardini, Gourieroux (2012)]. Indeed, we get from equation

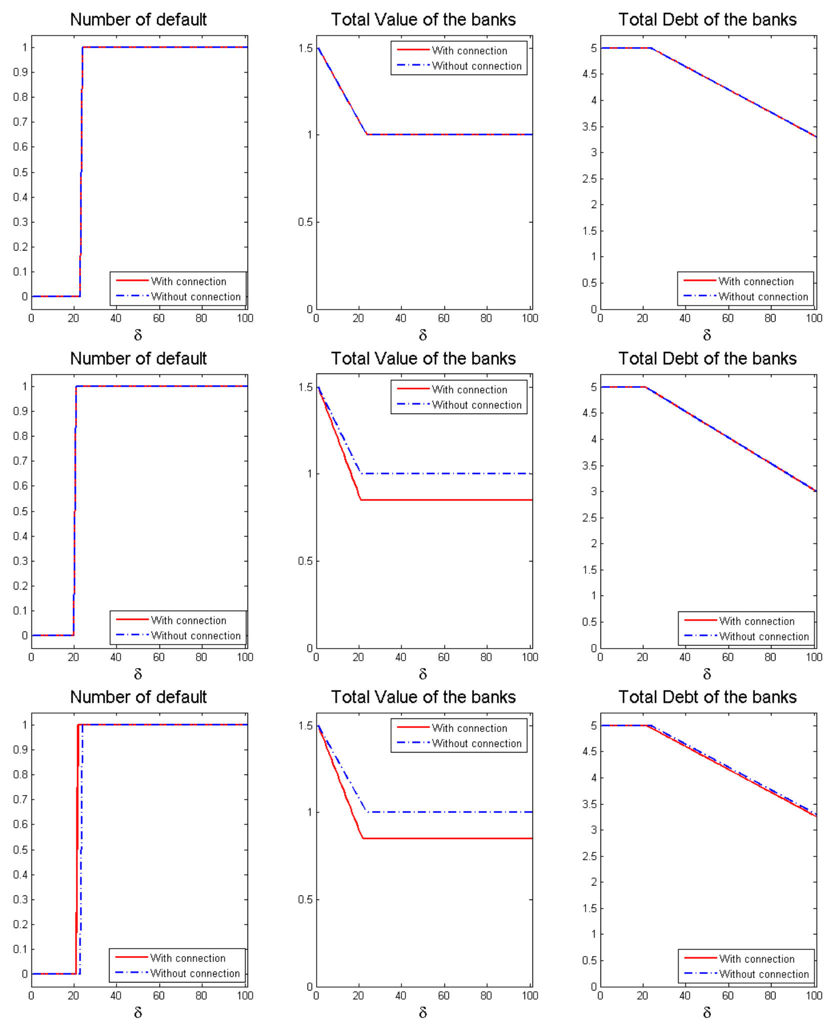

Figure 6: Decomposition of the Impulse Response Functions for Sets 1, 2 and 3

(4.5) :

$$\mathbb{V}(\mathbb{1}'\Delta Y) = \underbrace{\mathbb{V}(\mathbb{1}'\beta)}_{\text{Direct effect}} + \underbrace{\mathbb{V}\Big(\mathbb{1}'(\sum_{j=1}^{\infty}\Pi^j)\beta\Big)}_{\text{Contagion effect}} + \underbrace{2 \times \mathbb{C}ov\Big(\mathbb{1}'\beta; \mathbb{1}'(\sum_{j=1}^{\infty}\Pi^j)\beta\Big)}_{\text{Cross direct-contagion effect}}$$

$$= \mathbb{1}'\mathbb{V}(\beta)\mathbb{1} + \mathbb{1}'(\sum_{j=1}^{\infty}\Pi^j)\mathbb{V}(\beta)(\sum_{j=1}^{\infty}\Pi^j)'\mathbb{1} + 2 \times \mathbb{1}'\mathbb{V}(\beta)(\sum_{j=1}^{\infty}\Pi^j)'\mathbb{1}.$$

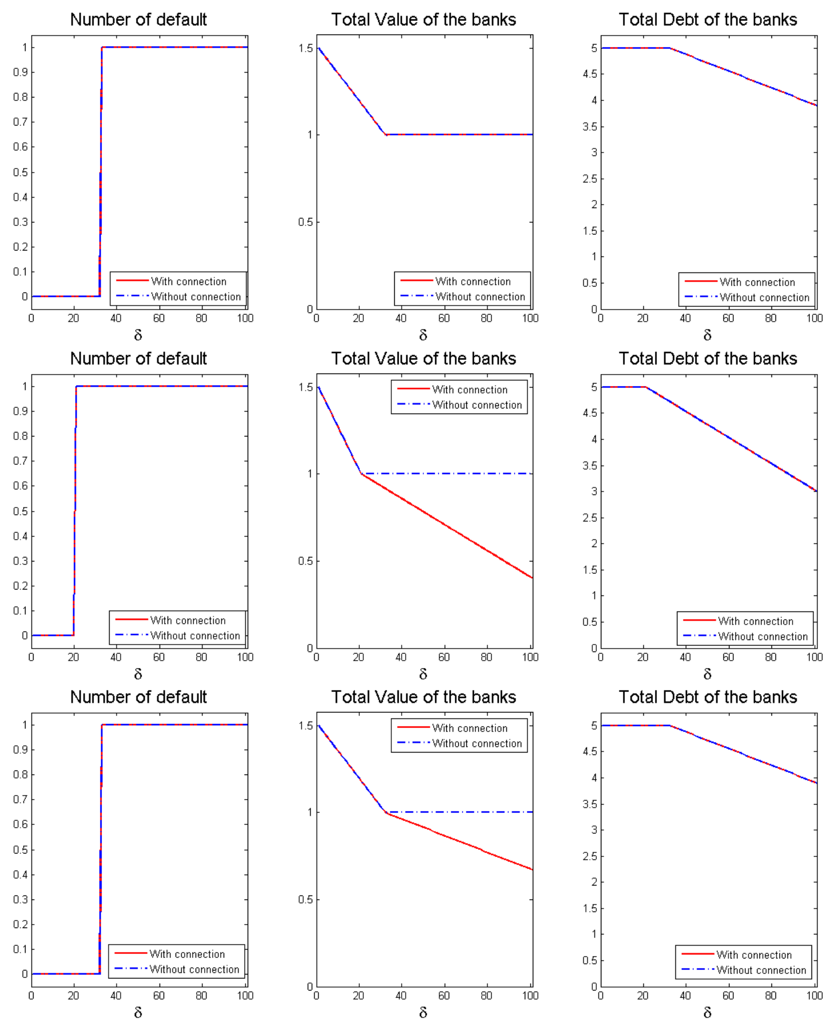

Figure 7: Decomposition of the Impulse Response Functions for Sets 4, 5 and 6

4.3 The effect of contagion on reverse stress-test

For a given bank, bank 1, say, and a shock specific to the exogenous asset component of this bank :

$$Ax = Ax^0 - \delta(Ax_1, 0, ..., 0)', \tag{4.10}$$

with $\delta \in (0,1)$, there is a minimal value of δ, which implies the first failure of a bank in the system. We might expect that this bank will be bank 1, but this is not always the case because of the stock interconnections. For instance, in a system of two banks with balance sheets such as :

- Bank 1 : $\pi_{1,1} = \pi_{1,2} = \gamma_{1,1} = \gamma_{1,2} = 0$, $L_1^* = 100$ and $Ax_1 = 200$,
- Bank 2 : $\pi_{2,1} = 50\%$, $\pi_{2,2} = \gamma_{2,1} = \gamma_{2,2} = 0$, $L_2^* = 100$ and $Ax_2 = 55$,

bank 2 will first fail for a decrease of 5% of Ax_1.

To illustrate the effect of contagion, we consider the French banking system as described above. For each bank i, we consider: i) the reverse stress-test of the initial system for a specific shock, ii) the reverse stress-test when the contagion effects are canceled, iii) the quantity $1 - L_i^*/Ax_i^0$. The levels of exogenous asset components and nominal debts at 12/31/2010 (in trillions of Euros) are :

$$Ax^0 = (0.58; 1.08; 1.59; 1.10; 1.98)', \quad L^* = (0.56; 1.09; 1.67; 1.08; 1.91)'.$$

The results of the reverse stress-tests are given in Table 3.

Specific shock on	Bank A	Bank B	Bank C	Bank D	Bank E
First bank to fail	Bank A	Bank B	Bank C	Bank D	Bank E
δ (with contagion, %)	5.810	4.544	3.073	4.559	4.340
δ (without contagion, %)	5.810	4.544	3.085	4.635	4.353
$1 - L_i^*/Ax_i^0$ (%)	1.77	-0.98	-5.36	1.76	3.19

Table 3: Reverse Stress-Tests for the Banking Sector (at 12/31/2010) ; δ in percent

Comparing the δ's with and without contagion, two cases arises. First, for two banks, there is no significative effect of contagion, since the δ's are equal. Second, the effect of contagion increases the δ for three banks. The difference in δ corresponds up to few hundreds millions of Euros, that is up to few percent of their prudential capitals.
A positive value in the last row means that the initial exogenous assets are sufficient to cover any possible loss on the debt and stock holding of the other

banks, whereas a negative value points out that at least a part of the interbank assets are needed to cover all the debt. Thus the solvency of two banks seems to be sensitive to the health of the banking sector. This is consistent with the larger row in the exposure matrices. The small contagion effects observed above are contingent to the selected system.

The optimal δ values in Table 3 admit explicit forms. Indeed, given the initial configuration $\mathcal{S}^0$ and a direction of shock with nonpositive components, we have :

$$\begin{aligned}\delta &= \max_d \Big\{ \text{there exists } i \text{ such that} \\ &\qquad \Big[(Id-\Pi)^{-1}((\Gamma - Id)L^* + Ax^0 + d\beta)\Big]_i = 0, \\ &\qquad \Big[(Id-\Pi)^{-1}((\Gamma - Id)L^* + Ax^0 + d\beta)\Big]_j > 0, \forall j \neq i \Big\} \\ &= \max_{i/\left[(Id-\Pi)^{-1}\beta\right]_i > 0} \Big\{ [(Id-\Pi)^{-1}((\Gamma - Id)L^* + Ax^0)]_i / [(Id-\Pi)^{-1}\beta]_i \Big\}.\end{aligned}$$

4.4 Decomposition of a probability of default

Let us consider the financial system $\mathcal{S}^0 = \{\Pi, \Gamma, L^*, Ax^0\}$ and a stochastic shock, that is a new exogenous asset component Ax which is stochastic. After the shock, we get the system : $\mathcal{S} = \{\Pi, \Gamma, L^*, Ax\}$. The equilibrium values $Y(\mathcal{S})$, $L(\mathcal{S})$ are stochastic [see Section 3.2.iii)]. In particular, we can compute the probability of default of bank i : PD_i.

As noted in Section 4.2, it is possible to cancel the contagion channel by considering the virtual initial financial system $\widetilde{\mathcal{S}^0} = \{0, 0, L^*, \widetilde{Ax^0}\}$, with $\widetilde{Ax^0} = \Pi Y^0 + \Gamma L^* + Ax^0 = Y^0 + L^*$. Then we can apply the same shock to system $\widetilde{\mathcal{S}^0}$ in order to get a virtual financial system after shock $\widetilde{\mathcal{S}} = \{0, 0, L^*, Ax - Ax^0 + \widetilde{Ax^0}\} = \{0, 0, L^*, Ax + \Pi Y^0 + \Gamma L^*\}$, and define the probability of default without contagion (or direct PD) as : PD_i^d.

A measure of the effect of contagion on the probability of default is :

$$K_i = \frac{PD_i}{PD_i^d}$$

Contagion has an increasing effect (resp. decreasing effect), if $K_i > 1$ (resp. $K_i < 1$).

As an illustration of the effect of stochastic shocks on the probability of default, we consider the same initial situation of the French banking system as in Section 4.3. Then, the stochastic shocks are introduced on the exogenous asset components as in the standard Vasicek extension of the Value-of-the-Firm model [Vasicek (1987)]. We assume that :

$$log(Ax_i) = log(Ax_i^0) + u_i, \quad i = 1, ..., n,$$

where the stochastic u_i's are Gaussian. We set $\mathbb{E}(u) = 0$, $\mathbb{V}(u) = \sigma^2 Id$, where $\sigma = 0.0141$.
The probability of default with and without contagion can be easily derived by simulation and can be converted into ratings. For instance, for this type of stochastic shock and without connection, the equivalent ratings vary from A to AAA [see e.g. Carey (2001), Table 1].
Considering the variation of individual probabilities of default in Table 4, the effect of interconnection is not uniform across banks. Generally speaking, being interconnected lowers the probability of default. The interconnections can be seen as an efficient diversification of risk since the stochastic shocks u_i's, are independent. The probabilities of joint default are reported Table 5, respectively without and with interconnections. Pairwise defaults appear slightly more with interconnections than without interconnections. Comparing this result to the general decrease of probability of default is consistent with previous results in the literature : interconnections can have ambiguous effects. Indeed, the numerical results relies on the selected shock.

	PD (in %) Without connection	PD (in %) With connection	ΔPD
bank A	0.001	0.000	-0.001
bank B	0.056	0.025	-0.003
bank C	1.348	1.391	+0.043
bank D	0.052	0.001	-0.041
bank E	0.091	0.002	-0.089

Table 4: Simulated Probabilities of Default for the Banking Sector (at 12/31/2010) ; 100,000 simulations

Without	Bank A	Bank B	Bank C	Bank D	Bank E
Bank A	0.001	0.	0.	0.	0.
Bank B		0.055	0.001	0.	0.
Bank C			1.346	0.	0.001
Bank D				0.042	0.
Bank E					0.090
With	**Bank A**	**Bank B**	**Bank C**	**Bank D**	**Bank E**
Bank A	0.	0.	0.	0.	0.
Bank B		0.024	0.001	0.	0.
Bank C			1.388	0.001	0.001
Bank D				0.	0.
Bank E					0.

Table 5: Simulated Probabilities of Joint Default for the Banking Sector (at 12/31/2010) ; 100,000 simulations ;

5 Concluding remarks

Until now very little was known about the actual structure of bilateral exposures in the Finance and Insurance sectors. The new regulations for financial stability require a periodic reporting by banks and insurance companies about their counterparties by class of assets, possibly distinguished by maturities and seniorities. This information might be used to quantify the bilateral exposures in terms of stocks, lendings, or derivatives. In our paper, we considered a simplified framework, which does not distinguish seniorities, maturities, and more generally liquidity features, and focus on solvency risk. We also exclude feedback effect through market prices of assets following banks' reactions to a shock as modeled in Greenwood, Landier and Thesmar (2012). This fire-sale phenomena can affect unconnected banks simply by their common exposure. We saw how such a structural information on the balance sheet can be used to define the system of banks and its structure after an exogenous shock, the so-called liquidation equilibrium. We also saw that this information can be used to decompose the systemic effect of an exogenous deterministic or stochastic shock into a direct and a contagion effect, respectively. Such a decomposition is appealing for the interpretation of stress tests and reserve for systemic risk. It is also appealing in a perspective of controlling systemic risk. Indeed an alternative to a control of systematic exogenous factors, such as the sovereign Greek debt for instance, is the control of the exposure matrices. Such a policy was followed by the Federal Reserve of New York to avoid the forced liquidation of LTCM. In order to avoid an uncontrolled transmission of losses from LTCM to its counterparties that could put the financial sector in distress, the FED asked several private banks to take control of this institution, that is, to change the matrix of bilateral stock exposures [see Greenspan(1998), McDonough(1998)].

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Appendix 1 Balance Sheet and Large Exposure European Templates

In order to collect data, the European National Supervisory Authorities could use the templates of the Committee of European Banking Supervisors (CEBS). These templates are filled by banks and controlled by National Supervisory Authorities.

Figures A.1.1 and A.1.2 are extracts of the balance sheet. The first one is the general structure of the asset side with in column : the financial item, the accounting rules (IFRS for International Financial Reporting Standards and IAS for International Accounting Standards), the reference to a sub-table where the amount is decomposed and, in last column, the amount. Figure A.1.2 is the sub-table for Financial Assets Held for Trading. The decomposition mixes financial instruments (Equity, Debt...) and nature of counterparties (Central Bank, General governments...). Besides the general accounting rules and the total marked-to-market amount, a decomposition between price and volume is given.

Figure A.1.3 is a table for large exposures report. In this template, the main counterparties (of the filling bank) are reported in column. The rows report the name of the counterparty, the total exposed amount, a breakdown of this amount across financial instruments, provision...

1. Consolidated Balance Sheet Statement [Statement of Financial Position]			
Table 1.1. Assets			
	References	**Breakdown in table**	**Carrying amount**
Cash and cash equivalents	*IAS 7.6-7, 45; IAS 1.54 (i)*		
Cash on hand			
Demand deposits and cash equivalents	*IAS 7.6-7, 45*	5	
Financial assets held for trading	*IFRS 7.8 (a)(ii); IAS 39.9, AG 14*	5	
Derivatives held for trading	*IAS 39.9*	3	
Equity instruments	*IAS 32.11*		
Debt securities			
Loans and advances			
Financial assets designated at fair value through profit or loss	*IFRS 7.8 (a) (i); IAS 39.9*	5	
Equity instruments	*IAS 32.11*		
Debt securities			
Loans and advances			
Available-for-sale financial assets	*IFRS 7.8.(d); IAS 39.9*	5	
Equity instruments	*IAS 32.11*		
Debt securities			
Loans and advances			

Figure A.1.1 : Extract from the main template for asset side [source : CEBS (2009)]

Table B. Financial assets held for trading			
	References	**Carrying amount**	**Amount of cumulative change in the fair values attributable to changes in the credit risk**
			IFRS 7.9 (c)
Equity instruments	*IAS 32.11*		
of which at cost			
of which: credit institutions			
of which: other financial corporations			
of which: non-financial corporations			
Debt securities			
Central banks			
General governments			
Credit institutions			
Other financial corporations			
Corporates			
Loans and advances			
Central banks			
General governments			
Credit institutions			
Other financial corporations			
Corporates			

Figure A.1.2 : Extract from an asset side decomposition template for Financial Assets Held for Trading [source : CEBS (2009)]

1	Code	Identification code for the counterparty or group based on national practices	...	...
2	Institution	"1" for non-credit institutions; "2" for credit institutions (counterparty meets the definition in Article 3(c) of Directive 2006/49/EC or Article 107 of Directive 2006/48/EC); "3" for intra-group credit institutions; "4" for intra-group non-credit institutions	2	2
3	Name	Name of the counterparty or "Group"	Bank A	Bank B
4	Exposure before risk provisioning: Total	5+6+7+8	100	50
5	Of which: Assets	Assets referred to in Article 74 of Directive 2006/48/EC not included in any other category.	100	10
6	Of which: Derivatives	Balance sheet value before value adjustments and provisions	0	0
7	Of which: Off-balance sheet	Off- balance sheet value before value adjustments and provisions	0	0
8	Of which: Indirect exposures	Article 117 paragraph 1 of Directive 2006/48/EC. Credit risk mitigation techniques that increase the exposure on the guarantor or third party via substitution. Indirect exsposures arising from credit derivatives	0	40
9	(-) Value adjustments and provisions	Sum of value adjustments and provisions for 5+6+7+8	0	0
10	Total exposure before CRM	4+9	100	50
11	Of which: Banking book	Share of 10 which belongs to the Banking book	100	50

Figure A.1.3 : Extract from the Large Exposure template [source : CEBS (2009)]

Appendix 2

Proof of Proposition 1

In the case of n banks, the 2^n regimes can be indexed by a sequence $z = (z_1, \ldots, z_n)$ of 0 and 1, where $z_i = 1$, if bank i defaults, $z_i = 0$, otherwise. Let us define the matrix $Q(z)$ as follows. The i^{th} column of matrix $Q(z)$ is the i^{th} column of Γ when $z_i = 1$, of Π when $z_i = 0$.

i) A preliminary lemma :

Lemma A.1 : If $\pi_{i,j} \geq 0, \gamma_{i,j} \geq 0, \forall i,j, \sum_{i=1}^{n} \pi_{i,j} < 1, \forall j, \sum_{i=1}^{n} \gamma_{i,j} < 1, \forall j$, then $det[Id - Q(z)] > 0, \forall z$.

Proof :
By the assumptions in Proposition 1, the matrices $Q'(z)$ have nonnegative coefficients, which sum up to a value strictly smaller than 1 per row. By applying Perron-Froebenius theorem, we deduce that the eigenvalues of $Q'(z)$, which are also equal to the eigenvalues of $Q(z)$ have a modulus strictly smaller than 1. Therefore the eigenvalues of $Id - Q(z)$ are either complex conjugates, or real positive, and their product equal to $det[Id - Q(z)]$ is strictly positive.

QED

In particular, the matrices $Id - \Pi$ and $Id - \Gamma$ are invertible.

ii) Existence and Uniqueness :

The first equation of system (3.2) can be rewritten as :

$$\begin{aligned} Y_i &= \Big[\sum_{j=1}^{n} \pi_{i,j} Y_j - \sum_{j=1}^{n} \gamma_{i,j} \Delta L_j + Ax_i - L_i + \sum_{j=1}^{n} \gamma_{i,j} L_j^*\Big]^+ \\ &= \Big[\sum_{j=1}^{n} \pi_{i,j} Y_j - \sum_{j=1}^{n} \gamma_{i,j} \Delta L_j + \Delta L_i + \Delta Ax_i\Big]^+, \end{aligned}$$

with $\Delta L_i = L_i^* - L_i$, $\Delta Ax_i = Ax_i - Ax_i^*$ and $Ax_i^* = L_i^* - \sum_{j=1}^{n} \gamma_{i,j} L_j^*$. The second equation of system (3.2) can be rewritten as :

$$\begin{aligned} \Delta L_i &= -min\Big[\sum_{j=1}^{n} \pi_{i,j} Y_j + \sum_{j=1}^{n} \gamma_{i,j} L_j + Ax_i - L_i^* \quad , \quad 0\Big] \\ &= \Big[- \sum_{j=1}^{n} \pi_{i,j} Y_j + \sum_{j=1}^{n} \gamma_{i,j} \Delta L_j - \Delta Ax_i \Big]^+. \end{aligned}$$

Therefore in regime z, the equilibrium values are such that :

$$z_i Y_i + (1 - z_i)\Delta L_i = 0, i = 1, \dots, n, \tag{a.1}$$

$$\begin{bmatrix} (1 - z_1)Y_1 - z_1 \Delta L_1 \\ \vdots \\ (1 - z_n)Y_n - z_n \Delta L_n \end{bmatrix} = [Id - Q(z)]^{-1} \Delta Ax. \tag{a.2}$$

Equations (a.1) say that $Y = 0$ for a defaulted bank and $\Delta L = 0$ for a non defaulted bank. Equations (a.1) and (a.2) provide the equilibrium values of ΔL for the defaulted banks and the equilibrium values of Y for the non defaulted banks.

We deduce that regime z occurs iff :

$$\Delta Ax \in [Id - Q(z)] \prod_{i=1}^{n} \Big\{(1 - z_i)(I\!R^+) + z_i[-L_i^*, 0]\Big\} = \mathcal{C}(z), \text{ say.}$$

The liquidation equilibrium exists for any admissible Ax iff the union of the truncated cones $\mathcal{C}(z)$, z varying, contains the set of admissible values of ΔAx. This condition is :

$$\bigcup_z \mathcal{C}(z) \supset -Ax^* + (I\!R^+)^n = -(Id - \Gamma)L^* + (I\!R^+)^n,$$

since the exogenous asset components Ax_i are positive. Note that, since $L^* \in (I\!R^+)^n$, Ax^* cannot belong to $(I\!R^-)^n$, because, in this case, $L^* = (Id - \Gamma)^{-1} Ax^* = (Id + \Gamma + \Gamma^2 + ...)Ax^*$ would belong to $(I\!R^-)^n$.
When it exists the liquidation equilibrium is unique iff the truncated cones $\mathcal{C}(z)$, z varying, do not overlap.

To analyze the existence and uniqueness of this equilibrium let us consider the piecewise linear function from $I\!R^n$ to $I\!R^n$

$$g(x) = \sum_z [Id - Q(z)] x \mathbb{1}_{[x \in \mathcal{C}^*(z)]}, \tag{a.3}$$

where $\mathcal{C}^*(z) = \prod_{i=1}^{n} \Big\{ (1-z_i) I\!R^+ + z_i I\!R^- \Big\}$ denotes the orthants of $I\!R^n$.
We have :

$$g(\mathcal{C}^*(z)) = \overline{\mathcal{C}}(z), \tag{a.4}$$

where $\overline{\mathcal{C}}(z)$ is the cone generated by the truncated cones $\mathcal{C}(z)$.
The proof of Proposition 1 is based on Theorem 1 in Gourieroux, Laffont, Monfort (1981), given below in our framework.

Theorem 1 : *The following properties are equivalent :*

i) Function g is one-to-one from $I\!R^n$ to $I\!R^n$;

ii) $det[Id - Q(z)]$, z varying, have the same sign ;

iii) $\bigcup_z \overline{\mathcal{C}}(z) = I\!R^n$;

iv) the cones $\overline{\mathcal{C}}(z)$, z varying, do not overlap.

i) First note that the truncated cones are non degenerate, i.e. reduced to $\{0\}$, and that they do not overlap iff their extensions $\overline{\mathcal{C}}(z)$ do not overlap. By Theorem 1, we deduce that a necessary and sufficient condition for the uniqueness of the liquidation equilibrium is : "$det[Id - Q(z)]$, z varying, have the same sign".

ii) Then, we have to check that the equivalent condition $\bigcup_z \overline{\mathcal{C}}(z) = I\!R^n$ implies the condition for existence :

$$\bigcup_z \mathcal{C}(z) \supset -Ax^* + (I\!R^+)^n,$$

with $Ax^* = (Id - \Gamma)L^*$. Since $\bigcup_z \overline{\mathcal{C}}(z) = I\!R^n$, the previous condition can be written :

$$\begin{aligned} \bigcup_z \mathcal{C}(z) &\supset \Big(\bigcup_z \overline{\mathcal{C}}(z) \Big) \cap \Big(-Ax^* + (I\!R^+)^n \Big) \\ &\supset \bigcup_z \Big(\overline{\mathcal{C}}(z) \cap (-Ax^* + (I\!R^+)^n) \Big) \end{aligned}$$

which is equivalent to :

$$\mathcal{C}(z) \supset \overline{\mathcal{C}}(z) \cap (- Ax^* + (I\!R^+)^n), \qquad \forall z,$$

(since $\mathcal{C}(z) \subset \overline{\mathcal{C}}(z)$ and the $\overline{\mathcal{C}}(z)$ do no overlap). If we denote by $M(z)$ the point in $I\!R^n$ with ith coordinate 0 if $z_i = 0$ and $-L_i^*$ if $z_i = 1$, we get :

$$\mathcal{C}(z) = \overline{\mathcal{C}}(z) \cap \Big([Id - Q(z)][M(z) + (I\!R^+)^n] \Big).$$

Moreover $[Id - Q(z)](I\!R^+)^n \supset (I\!R^+)^n$ since any point y of $(I\!R^+)^n$ is the image by $Id - Q(z)$ of $[Id - Q(z)]^{-1}y = y + Q(z)y + Q^2(z)y + ...$ which belongs to $(I\!R^+)^n$, and we get

$$\mathcal{C}(z) \supset \overline{\mathcal{C}}(z) \cap \Big([Id - Q(z)]M(z) + (I\!R^+)^n \Big).$$

We now have to check that $\overline{\mathcal{C}}(z) \cap \Big([Id - Q(z)]M(z) + (I\!R^+)^n \Big) \supset \overline{\mathcal{C}}(z) \cap (- Ax^* + (I\!R^+)^n)$ for any z. For instance, for $z = (1, ..., 1)' = e$, we have $[Id - Q(e)]M(e) = -Ax^*$ and the result holds.

iii) Finally, from Lemma A.1 above, the determinants of $Id - Q(z)$ have the same positive sign under the assumptions of Proposition 1 and the results of this proposition follow.

Appendix 3

Proof of Proposition 2

The equilibrium values $\begin{pmatrix} Y \\ -\Delta L \end{pmatrix}$ can be written as :

$$\begin{pmatrix} Y \\ -\Delta L \end{pmatrix} = \begin{pmatrix} Y \\ L - L^* \end{pmatrix} = \Sigma_z \{ D^*(z)(Ax - Ax^*) \mathbb{1}_{Ax - Ax^* \in \mathcal{C}_z} \}, \qquad \text{(a.5)}$$

where $D^*(z)$ is a $(2n \times n)$ matrix built as follows : for the first n rows, the i^{th} row is null if $z_i = 1$, for the last n rows, the i^{th} row is null if $z_i = 0$ and the nonzero rows are exactly the rows of $D(z)^{-1}$. Since, $\begin{pmatrix} Y \\ -\Delta L \end{pmatrix}$ is a continuous piecewise linear function of Ax, we have just to check that the function is nondecreasing in each regime $\mathcal{C}_z$

After an appropriate permutation of variables, the nonzero block of the $D^*(z)$ matrix has the form $[Id - Q(z)]^{-1}$, say, where $Q(z)$ is a matrix with nonnegative coefficients, which sum to a value strictly smaller than 1 per column. As an illustration the $Q(z)$ matrices for the case of two banks $n = 2$ are :

regime of no default : $Q(0,0) = \Pi$,

regime of joint default : $Q(1,1) = \Gamma$,

regime of default of bank 1 : $Q(1,0) = \begin{pmatrix} \gamma_{1,1} & \pi_{1,2} \\ \gamma_{2,1} & \pi_{2,2} \end{pmatrix}$.

and a similar expression for the regime of default of bank 2:
$Q(0,1) = \begin{pmatrix} \pi_{1,1} & \gamma_{1,2} \\ \pi_{2,1} & \gamma_{2,2} \end{pmatrix}$. It is easily checked that the eigenvalues of $Q(z)$, equal to the eigenvalues of $Q'(z)$, have a modulus strictly smaller than one by Perron-Froebenius theorem (see Lemma A.1 in Appendix 1), and, using

the expansion :

$$[Id - Q(z)]^{-1} = \sum_{h=0}^{\infty} [Q(z)^h],$$

we deduce that this inverse matrix has nonnegative elements, since the elements of $Q(z)^h$ are nonnegative for any h.

Appendix 4

Merton's Model

Merton (1974) presents a stylized approach for evaluating credit risk of a single firm. Let us summarize and discuss the main features of this paper.

a) Merton's model

The firm has the following simple balance sheet :

Asset	Liability
Ax	L^*

Ax includes all the assets of the firm while its nominal debt is L^*. Besides, Merton identifies two types of stakeholders : the shareholders and the debtors. The shareholders own the value of the firm Y while the debtors hold its debt L of nominal value L^*.

Based on these elements, Merton derives the pricing of those components with the respect to the status of the firm : either default, or alive. The status is triggered by the relative value of asset Ax over nominal debt L^*. We get :

$$\begin{cases} Y = (Ax - L^*)^+, \\ L = min(Ax, L^*). \end{cases} \tag{a.6}$$

b) Equilibrium conditions and solutions

Let us now consider the system :

$$\begin{cases} Y = (Ax - L)^+ \\ L = min(Ax, L^*) \end{cases}$$

where Y and L are simultaneously defined. The equilibrium solution of this new system is exactly the quantity given in $(a.8)$.

Therefore, the standard Merton's model is a special system with a single bank and no self-holding of both stocks and lendings.

c) Interpretation in terms of options

A standard interpretation of Merton's model is to consider that shareholders buy a call on Ax with strike L^*, whereas debtors sell a portfolio including a put on Ax with strike L^* and risk free asset. This interpretation is summarized in the following table :

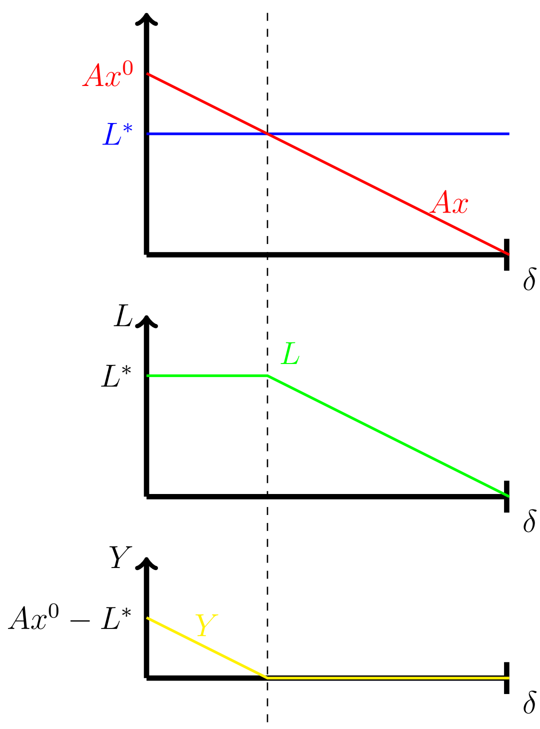
	Stakeholder	
Status	Shareholder	Debtor
$Ax > L^*$	$Y = Ax - L^*$	$L = L^*$
$Ax \leq L^*$	$Y = 0$	$L = Ax$
	$Y = (Ax - L^*)^+$	$L = min(Ax, L^*)$
	buying a call	selling a put

d) Convexity property

L is concave in Ax. Since Y is convex in Ax, $-Y$ is concave in Ax.

e) Impulse response

Consider an initial situation where $Ax = Ax^0$. We plot below the evolutions of L and Y as Ax decreases through a factor δ down to zero value. They show the convexity (resp. concavity) property of Y (resp. L) as a function of δ.



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